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SOLUTIONS OF THE EXAMPLES in

DIFFERENTIAL CALCULUS

A FIRST COURSE
IN
VECTOR ANALYSIS

BY

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PREFACE

The present treatise on Vector Analysis is intended to be a text-book for undergraduate classes in Indian Universities. The notion of a vector has been approached from two points of view—Geometric and Algebraic. A correspondence between the two has also been established. It has been the aim of the authors to make available to the reader a tool of which he may make a profitable use in his studies of the various branches of Mathematics, and of Analytic Geometry and Mechanics, in particular.

We take this opportunity to acknowledge with gratitude the help we have derived from standard treatises on the subject. We shall consider our labour amply rewarded if our contributions through this book render help to the students for whom it is meant. Critical suggestions for improvement of the book will be thankfully received.

We express our heartfelt gratitude to Pratul Kumar Bagchi, Professor of Mathematics, Vidyasagar College, Calcutta, who has been a perennial source of inspiration to us in the preparation of this book.

Our thanks in preparing this book are also due to Rev. Father Goreux. S. j. ; D. Sc. of the St. Xavier's College, Calcutta, Profs. B. Haldar, J. N. Dasgupta of the Asutosh College, Calcutta, Prof. G. C. Chakravarty of the Vidyasagar College, Calcutta, Prof. M. C. Ghosh of the Dinabandhu Andrews College, 24-Parganas, Prof. K. R. Roy of the Presidency College, Calcutta, Prof. T. P. Maity of the Scottish Church College, Calcutta, Prof. S. K. Sur of the Charu Chandra College, Calcutta and Prof. A. Chakravarty of the P. K. College, Contai, for advice and suggestions of various sorts.

CALCUTTA,
25th. November, 1958.

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PREFACE TO THE SECOND EDITION

A second edition of *A first course in Vector Analysis* has been called for within a remarkably short time. This has been a definite encouragement to the authors who take this opportunity to extend their sincerest thanks to the teachers and students of different Colleges and Technical Institutes situated in various parts of India for according a ready welcome and favourable reception to the book.

In this edition the general plan of the book has not been changed. But we have attempted to satisfy the long-felt need for a short course on Vector Calculus suitable for undergraduate Honours students. Beginning with the notion of vector functions of a single scalar variable, we have introduced systematically the concepts of limit, continuity, derivability, indefinite integration as the reverse of differentiation, definite integration as the limit of an infinite sum. Applications of these concepts in Differential Geometry of Curves and Mechanics (Kinematics, in particular) with profuse illustrations have been furnished.

This preface would be incomplete if we close it without an expression of deep gratitude which we owe to Dr. N. N. Bose of Lucknow Christian College, Lucknow, Vice-Principal M. R. Das Gupta of City College, Calcutta, Prof. G. D. Bhar of St. Paul's College, Calcutta, Dr. K. N. Bhattacharya of Presidency College, Calcutta, Prof. B. C. Dam and Prof. T. M. Mukherjee of Vivekananda College, Barisha, Dr. D. N. Mitra of I. I. T., Kharagpur, for their valuable suggestions and kind encouragement.

CALCUTTA
4th. July, 1960

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PREFACE TO THE THIRD EDITION

This is an extensive revision of the previous editions of this book. Most of the matter has been re-written with greater attention to clarity. The number of exercises has been increased considerably. All the diagrams have been drawn afresh. Answers with sufficient hints are provided in most of the exercises at the end of each example.

For a quick recapitulation of new concepts, the summary of two useful sections *viz.*, chapters 1 and 3 have been appended.

In course of developing the book in the present form we have taken advantage of the suggestions from many of our fellow-teachers to all of whom we extend our sincere gratitude. We are also thankful to those who have drawn our attention to some errors which had crept in the preparation of the previous editions. Suggestions for further improvement will be cordially received.

CALCUTTA
The 15th. August, 1962.

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1. Elementary Operations : Centroids

2. Applications of Elementary operations : Geometrical and Physical

3. Product of Vectors

4. Applications on Vector products

5. Differentiation and Integration of Vectors

6. Elements of Differential Geometry

1

Elementary Operations : Centroids

1·1. Introduction.

The physical objects *i.e.*, measurable objects of reasoning in Mathematics are primarily of two distinct classes—those which do not involve the idea of direction are called *Scalars* (or *Scalar quantities*) and those which do involve the idea of direction are called *Vectors* (or *Vector quantities*).

Numbers, time, mass, length, temperature, quantity of heat, numerical statistics (*e.g.*, birth rates, mortality, population), electric charge, potential etc.—all possess magnitude *without any reference to direction*. They are *Scalar quantities*.

Displacement, velocity, acceleration, force, electric current, magnetic flux, lines of force, stress and strain due to any cause, flow of heat and fluids—all involve two ideas viz. *magnitude* and *direction*. They are *Vector quantities*.

1·11. Scalar quantities.

Scalar quantities are characterised by their magnitudes only without any reference to direction. In order to specify such a quantity we need to choose some unit quantity of the same kind and find how many such units are contained in a given quantity. Thus if u denotes the unit of such a quantity and m, the number of units contained, the given quantity is completely known by the expression mu. The ratio m which a given quantity bears to the chosen unit is called the *measure* or *magnitude* of the given scalar, quantity. Accordingly, *real numbers* are included among the scalars and are the simplest of them all. In what follows we shall consider a *real number* to represent some scalar ;

thus our scalars will obey all the laws of algebraic analysis of real numbers. The specification of a scalar by a real number permits of its geometric representation ; for, if we agree to represent u, the chosen unit by a line of definite length, then mu would be represented by a line m times as long as the line for the unit. The length of the line representing u will be called the *Scale of representation.*

Note. The name *Scalar* is significant because they may be specified by numbers marked off on a chosen scale. Latin *Scalae'* means a ladder, divided into parts by the rungs.

1·12. Directed line Segment.

Let an indefinite straight line L be given and let two points, A and B be marked off on L. Then the portion of L which is bounded by A and B is called a *line segment*. We may consider the line segments AB and BA as different segments by giving each a *direction* or *sense*. Thus AB shall be directed from A to B and BA shall be directed from B to A *i.e.*, oppositely to AB. These line segments are then called *directed line segments*. We denote them by $\overrightarrow{AB}$ and $\overrightarrow{BA}$, to distinguish them from the undirected line segments AB or BA (Fig. 1.1). In $\overrightarrow{AB}$ we call A the *initial point* and B the *terminal point* and in $\overrightarrow{BA}$ we say B, the initial point and A, the terminal point.

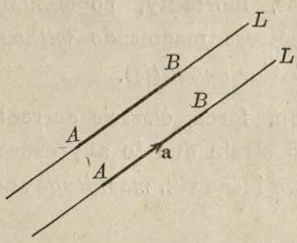


Fig. 1.1. Directed and undirected segments

With every directed line segment we attribute three characteristics :

(i) Length ; (ii) Support ; (iii) Sense.

(i) Length. The length of the directed line segment AB is the length of the line segment AB, *a scalar*. The notation for

this length is $|\overrightarrow{AB}|$; read : *absolute value of* $\overrightarrow{AB}$.

Note that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$.

(ii) Support. The support of $\overrightarrow{AB}$ is the line L of indefinite length of which the AB is a portion.

(iii) Sense. The sense of $\overrightarrow{AB}$ is indicated by the order in which the letters are stated *i.e.*, *from A towards B*.

If for two directed line segments *one* or *more* of these three characteristics differ they are treated as *different segments*.

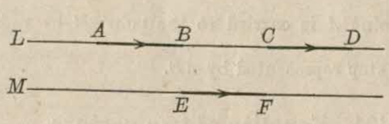


Fig. 1.2. Equality of directed segments

Note 1. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have same length, same support but of *opposite sense* ; hence they are different (directed) line segments. We call them *equal and opposite* directed segments.

Note 2. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (Fig. 1.2) have same support and same sense. Now if $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ then they will be treated as *equal directed segments*.

Note 3. $\overrightarrow{AB}$ and $\overrightarrow{EF}$ (Fig. 1.2) have parallel supports L and M. If now $|\overrightarrow{AB}| = |\overrightarrow{EF}|$ we may treat them as *equal segments* in the wider sense of the term. If two segments are not on the *same* or *parallel* supports they can never be equal.

1·2. Vector : free and localised.

A vector is a directed line segment. A vector from A to B (Fig. 1.1) is denoted by $\overrightarrow{AB}$, where it is necessary to specify its end points otherwise, by a single bold face letter, **a**, **b**, **c**...... or by some Greek alphabet α, β, γ...... .

The length of the vector $\overrightarrow{AB}$ or **a** is denoted by $|\overrightarrow{AB}|$ or by $|\mathbf{a}|$ (or simply by a) ; read : *length of the vector* $\overrightarrow{AB}$ (or **a**)

or *absolute value of* $\overrightarrow{AB}$ (or **a**) ; $|\mathbf{a}|$ is also known as the *measure* or *module* or *magnitude* of the vector **a**. In practice, the students may indicate a vector by a single letter with an arrow above it (*e.g.*, $\vec{a}$).

Note. The concept of vector is closely associated with displacement. In fact, *vector* means *carrier*, that which carries or displaces a point from one position to another (Latin *veho—I carry*). The terms *step, stroke, directed numbers* are synonymous with the term *vector*. We may interpret : point A is carried to the point B by means of a carrier $\overrightarrow{AB}$ or by following a step represented by $\overrightarrow{AB}$.

1·21. Equality of two vectors.

DEFINITION. Two vectors **a** and **b** are equal, written as $\mathbf{a}=\mathbf{b}$, if they have

(i) same length *i.e.*, $|\mathbf{a}|=|\mathbf{b}|$;

(ii) same sense ;

(iii) same or parallel supports.

Thus in Fig. 1.2, $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ are equal vectors.

Note carefully that the equality of two vectors does not depend on the absolute positions of their ends. In other words, our definition of equality does not demand that the vectors must have the *same* supports. The vectors conforming to such a definition of equality are said to be *free*.

A *Free vector* is thus a directed line segment occupying any position in space ; its initial point may be chosen arbitrarily on parallel supports.

In some applications, however, we meet with vectors which are restricted to lie on a given line (*e.g.*, forces acting on a rigid body are restricted to lie along their lines of action ; shifting along other lines will alter their dynamical effect). Such a

vector which is confined to a definite line of support is called a *Line vector.* Two such line vectors are equal if their *lengths, supports* and *senses* are same.

Note that two directed line segments may be *equal free vectors* but *unequal line vectors.* If *free,* then $\overrightarrow{AB} = \overrightarrow{EF}$ (Fig. 1.2), but they are not equal if they are *line vectors.*

In our discussions we shall mean a *free vector* when the term *vector* will be used. In case we need to consider a *line vector* we shall specifically mention it to be so.

1·22. Vectors and vector quantities.

We have defined the term *Vector* from a purely geometric concept (*directed line segment*) ; some prefer to call such a vector as a *length vector.* It is our object to develop an Algebra of length vectors. In order to make a study of the *vector quantities* (like velocity, acceleration etc.) we first require to fix up their units and then length vectors are used to specify their magnitudes and directions.

Again some vector quantities are *free* (*e.g.*, the translation of a rigid body) and others are *localised* (*e.g.*, forces acting on rigid body require for their specifications the knowledge of magnitude, direction as well as the definite positions they occupy in space). It is to be noted that a *single free vector* (length-vector) can not completely represent the effect of a localised vector quantity (*e.g.* two free vectors are necessary to specify a force acting on a rigid body).

The idea behind introducing Vector Analysis based on the concept of length-vectors is that, we may conveninetly apply the laws deduced here not only in problems of vector quantities but also in the study of geometrical problems (three-dimensional geometry, in particular). In the present treatise we have given more stress on the latter study.

We shall make use of the term *Vector Algebra* by which we shall mean a set of rules which are gainfully employed in combining a vector with another vector or a vector with a scalar ; the rules that are set are known as *Laws of combination of vectors.*

In the later chapters we shall also include a brief discussion of *Vector Calculus* where the concept of differentiation and integration of a vector function with respect to a scalar variable will be introduced. Our study will thus comprise the following :

1. Addition and subtraction of vectors.
2. Multiplication of vectors by scalars.
3. Different ways of multiplication of two vectors.
4. Derivation and integration of a vector function with respect to a scalar variable.
5. Applications in geometry and mechanics.

1·23. Definitions of a few important terms.

(A) Unit Vector : A vector whose length is unity is a *unit vector.* Unit vector thus indicates the direction. Unit vector of $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$, gives the direction of $\mathbf{a}$, length being equal to 1. Because of this directional property Heaviside used the term ORT (short for *orientation*) for a unit vector.

(B) Zero Vector : A *zero vector* (also called a *null vector*) is a vector whose length is zero. Its initial point and terminal point coincide. Thus.

$$\overrightarrow{AA}=\mathbf{0},\ \overrightarrow{BB}=\mathbf{0},\ \text{etc.}$$

Note that we have used a bold-face type zero to denote a null vector. Further, see that all null vectors are equal because any line may be considered as the line of support of a null vector.

Note. We shall use 0 to represent the number zero and $\mathbf{0}$ to represent a null vector. This double use of the same symbol is suggestive because both have many properties in common.

(C) Proper Vector : Every vector which is not a null vector is called a *proper vector*. When $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a}$ is a proper vector.

(D) Co-initial Vectors : Two or more vectors having the same initial point form *co-initial vectors*. Our vectors being free we may shift them in such a way that they may have the same initial point.

(E) Collinear Vectors : Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *collinear* (or *like* or *parallel*) when they have the *same* or *parallel* supports.

Thus in Fig. 1.2, $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ are collinear.

Note that two (free) vectors are equal when they are collinear as well as of same length and sense.

(F) Coplanar Vectors : A system of vectors is said to be *coplanar* if their supports are parallel to the same plane ; otherwise they are *non-coplanar*. A plane parallel to the system of coplanar vectors is called the *plane of the vectors*. Co-initial vectors need not be coplanar. A plane can always be drawn parallel to two given vectors.

1·3. Addition of vectors : Triangle law.

A rectilinear displacement or a translation from A to B may be represented by the vector $\overrightarrow{AB}$. If a particle be given two displacements successively, one from A to B and a second from B

to C, the result is the same as if the particle were given a single displacement from A to C. This suggests that

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}.$$

We shall regard this suggestion as the definition of vector addition.

Triangle law : *Given two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *; draw* $\mathbf{b}$ *from the terminal point of* $\mathbf{a}$, *then the vector directed from the initial point of* $\mathbf{a}$ *to the terminal point of* $\mathbf{b}$ *is called the sum of* $\mathbf{a}$ *and* $\mathbf{b}$, *written as* $\mathbf{a}+\mathbf{b}$.

This method of constructing a triangle in order to define the sum $\mathbf{a}+\mathbf{b}$ is called *Triangle law of addition of two vectors.*

In Fig. 1.3, $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$; then complete the triangle ABC ; $\overrightarrow{AC}$ will now represent the sum $\mathbf{a}+\mathbf{b}$.

Observe that the relation

$$AC \leqslant AB+BC$$

of plane geometry gives

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}|+|\mathbf{b}| ;$$

the equality holds when $\mathbf{a}$ and $\mathbf{b}$ have the same direction.

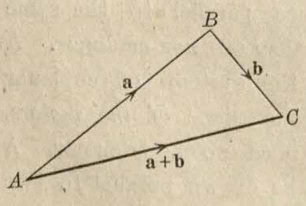


Fig. 1.3. Triangle law of addition

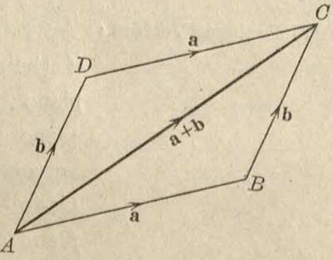


Fig. 1.4. Parallelogram law

Parallelogram law of addition : Completing the parallelogram $ABCD$ with AB and BC as sides, we note (Fig. 1.4)

$$\overrightarrow{AD}=\overrightarrow{BC}=\mathbf{b}, \qquad \cdots \quad (1)$$

($\because$ their supports are parallel, lengths and senses are same).

Hence, $\overrightarrow{AC} = \mathbf{a} + \mathbf{b} = \overrightarrow{AB} + \overrightarrow{BC}$ (Triangle law)

$$= \overrightarrow{AB} + \overrightarrow{AD} \quad (\text{by } (1))$$

Thus the sum of two co-initial vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is given by $\overrightarrow{AC}$, where AC is the diagonal of the parallelogram $ABCD$ having AB and AD as adjacent sides. This is called *Parallelogram law of addition of two vectors.*

Special Case : Since $\overrightarrow{AA} = \mathbf{0}$; $\overrightarrow{BB} = \mathbf{0}$, we may write

$$\mathbf{a} + \mathbf{0} = \overrightarrow{AB} + \overrightarrow{BB} = \overrightarrow{AB} = \mathbf{a}$$

or, $$\mathbf{0} + \mathbf{a} = \overrightarrow{AA} + \overrightarrow{AB} = \overrightarrow{AB} = \mathbf{a}.$$

These give properties of a null vector similar to the properties of the number 0 $(a + 0 = 0 + a = 0)$.

1·31. Addition of several vectors : Vector Polygon.

Continuing the argument already achieved for the addition of two vectors we may easily extend the method of construction for addition of a number of vectors **a, b, c, d, e, f** (say).

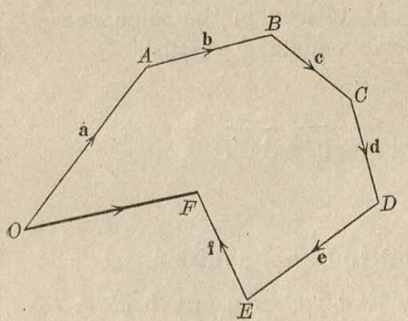


Fig. 1.5(*a*). Vector polygon

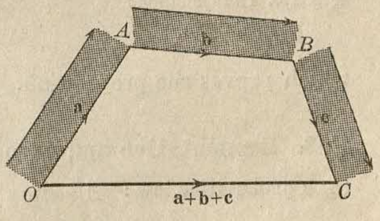


Fig. 1.5(*b*). Vectors need not be coplanar

Thus from Fig. 1.5(*a*) we have

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e} + \mathbf{f}$$
$$= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$$
$$= \overrightarrow{OF}.$$

The vector $\overrightarrow{OF}$ is then called the *sum* and the polygon $OABCDEF$, the *Vector Polygon*. Note that the rule for addition of several vectors does not require that the vectors should be coplanar [see Fig. 1.5(*b*)]

1·32. The sum of two vectors is independent of the choice of the initial point of the first vector.

Let A, A' be any two points in space (Fig. 1.6). Suppose,

$$\overrightarrow{AB}=\mathbf{a},\ \overrightarrow{BC}=\mathbf{b},\text{ so that }\overrightarrow{AC}=\mathbf{a}+\mathbf{b}.$$

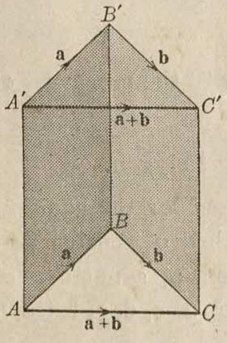


Fig. 1.6. Independence of initial point

Draw $A'B'$ and $B'C'$, equal and parallel to AB and BC respectively. Then

$$\overrightarrow{A'B'}=\mathbf{a},\ \overrightarrow{B'C'}=\mathbf{b}.$$

It can now be easily shown, by Geometry, that $\overrightarrow{A'C'}=\overrightarrow{AC}$; for, the lines joining the extremities of two equal and parallel lines (drawn in the same sense) are themselves equal and parallel.

Thus,

$$\mathbf{a}+\mathbf{b}=\overrightarrow{AC}=\overrightarrow{A'C'},$$

which proves the proposition.

1·33. Commutative and Associative law of addition.

(A) Commutative law : *Addition of two vectors* **a** *and* **b** *obeys the Commutative law, i.e.,*

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}.$$

Proof. We refer to Fig. 1.4. Let $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$. Then, by triangle law of addition,

$$\overrightarrow{AC}=\mathbf{a}+\mathbf{b}.$$

Complete the parallelogram $ABCD$. Then

$$\overrightarrow{AD}=\overrightarrow{BC}=\mathbf{b} \text{ and } \overrightarrow{DC}=\overrightarrow{AB}=\mathbf{a}.$$

From $\triangle ADC$, we now have

$$\mathbf{b}+\mathbf{a}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}=\mathbf{a}+\mathbf{b}.$$

(B) Associative law : *Vector addition is independent of the way its elements are associated in groups.*

Thus if **a, b, c** be any three vectors, then

$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}.$$

Proof. We refer to Fig. 1.7 where,

$$\overrightarrow{AB}=\mathbf{a},\ \overrightarrow{BC}=\mathbf{b},\ \overrightarrow{CD}=\mathbf{c}.$$

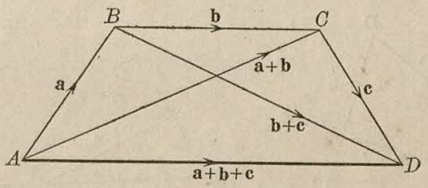


Fig. 1.7. Associative law

From $\triangle ABC$, $\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{a}+\mathbf{b}=\overrightarrow{AC}$

From $\triangle ACD$, $\overrightarrow{AC}+\overrightarrow{CD}=(\mathbf{a}+\mathbf{b})+\mathbf{c}=\overrightarrow{AD}$ ... (1)

From $\triangle BCD$, $\overrightarrow{BC}+\overrightarrow{CD}=\mathbf{b}+\mathbf{c}=\overrightarrow{BD}$

From $\triangle ABD$, $\overrightarrow{AB}+\overrightarrow{BD}=\mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{AD}$ ... (2)

Thus (1) and (2) prove the law. Since $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$, each side may simply be written as $\mathbf{a}+\mathbf{b}+\mathbf{c}$. Using commutative law, we may also write

$$(\mathbf{b}+\mathbf{c})+\mathbf{a}=\mathbf{c}+(\mathbf{a}+\mathbf{b})==\mathbf{a}+\mathbf{b}+\mathbf{c}.$$

Thus the sum of three vectors is independent of the order in which they are added and of their grouping to form partial sums.

Similar statements may be advanced for any number of vectors if we use successively the commutative and associative laws.

Note 1. Real numbers obey these two laws. If a, b, c be any three real numbers, then $a+b=b+a$; $a+(b+c)=(a+b)+c$.

Note 2. In constructing the vector sum of a number of vectors we are to place them successively ; the *sum* will be the vector directed from the initial point of the first to the terminal point of the last, *the order of succession being immaterial.*

1·4. Negative of a vector.

If x and y be two numbers such that $x+y=0$ then we say that *x is the negative of y* or *y is the negative of x.*

In a similar manner we define,

$\mathbf{a}$ *is negative of* $\mathbf{b}$ or *vice versa, if* $\mathbf{a}+\mathbf{b}=\mathbf{0}$.

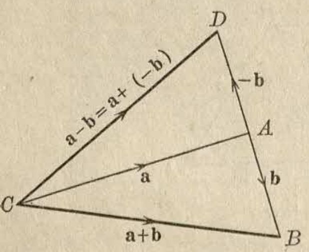


Fig. 1.8. Vector subtraction

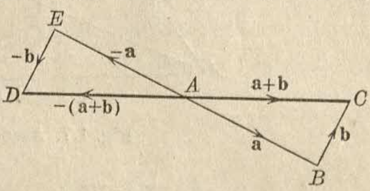


Fig. 1.9. Negative of $\mathbf{a}+\mathbf{b}$

From triangle law, $\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\mathbf{0}$. Hence $\overrightarrow{AB}$ is the negative of $\overrightarrow{BA}$ or *vice versa.* We agree to write $\overrightarrow{BA}=-\mathbf{a}$ if $\overrightarrow{AB}=\mathbf{a}$. The negative of $\overrightarrow{AB}$ is thus a vector $\overrightarrow{BA}$ of *same length, same support* but of *opposite sense.*

Note also that $\quad -(-\overrightarrow{AB})=\overrightarrow{AB}.$

1·5. Subtraction of two vectors.

In the number system we define subtraction as an inverse operation to addition. Thus to subtract a number b from a we

are to seek a new number c which when added with b gives a; c is then the *difference* $a-b$. We give similar definition to vector subtraction.

The difference of two vectors **a** *and* **b** (written as, $\mathbf{a}-\mathbf{b}$) *is a vector* **c** *which when added to* **b** *gives the sum* **a**.

Symbolically, $\mathbf{a}-\mathbf{b}=\mathbf{c}$, if $\mathbf{a}=\mathbf{c}+\mathbf{b}$.

Conversely, observe that $\mathbf{a}=\mathbf{c}+\mathbf{b}$ gives $\mathbf{a}=(\mathbf{a}-\mathbf{b})+\mathbf{b}$.

Hence adding $(-\mathbf{b})$ to both sides,

$$\mathbf{a}+(-\mathbf{b})=(\mathbf{a}-\mathbf{b})+\mathbf{b}+(-\mathbf{b})=(\mathbf{a}-\mathbf{b})+\mathbf{0}=\mathbf{a}-\mathbf{b}.$$

Hence subtraction of **b** from **a** is the same as adding the negative of **b** with **a**. This fact may be used in constructing $\mathbf{a}-\mathbf{b}$ (see Fig. 1.8).

It also follows $-(\mathbf{a}+\mathbf{b})=-\mathbf{a}-\mathbf{b}=(-\mathbf{a})+(-\mathbf{b})$ [Fig. 1.9]

1·51. Vector equation.

If **a**, **b** be two given vectors, then

$$\mathbf{a}+\mathbf{x}=\mathbf{b} \qquad \cdots \quad (1)$$

has a unique solution

$$\mathbf{x}=\mathbf{b}-\mathbf{a}. \qquad \cdots \quad (2)$$

To prove that (2) is a solution.

Substitute **x** given by (2) in (1). Then its left side becomes

$$\mathbf{a}+(\mathbf{b}-\mathbf{a})=\mathbf{a}+[(-\mathbf{a})+\mathbf{b}]=[\mathbf{a}+(-\mathbf{a})]+\mathbf{b}$$
$$=\mathbf{0}+\mathbf{b}=\mathbf{b}=\text{right side.}$$

To prove that the solution (2) is unique.

If possible, let **c** be another solution, then from (1)

$$\mathbf{a}+\mathbf{c}=\mathbf{b}$$

or, $-\mathbf{a}+(\mathbf{a}+\mathbf{c})=-\mathbf{a}+\mathbf{b}$ (add $-\mathbf{a}$ to both sides)

or, $(-\mathbf{a}+\mathbf{a})+\mathbf{c}=-\mathbf{a}+\mathbf{b}$ (Associative law)

or, $\mathbf{0}+\mathbf{c}=\mathbf{b}-\mathbf{a}$ (Commutative law)

or, $\mathbf{c}=\mathbf{b}-\mathbf{a}.$

Note that the existence of a solution of (1) is tacitly assumed. Thus in a vector equation the transfer of one vector from one side to the other can be made exactly as in ordinary Algebra of numbers.

1·6. Multiplication of a vector by a scalar.

DEFINITION. The product of a vector $\mathbf{a}$ by a scalar m, written as $m\mathbf{a}$, or $\mathbf{a}m$, is a *vector* whose

(i) *length* is $|m|$ times that of $\mathbf{a}$;

(ii) *support* is same or parallel to that of $\mathbf{a}$;

(iii) *sense* is same or opposite to that of $\mathbf{a}$ according as m is positive or negative.

In accordance with this definition we may define division of a vector $\mathbf{a}$ by a non-zero scalar m as the multiplication of $\mathbf{a}$ by the scalar $1/m$. Thus $\mathbf{a}/m=(1/m)\,\mathbf{a}$.

1·61. Some simple deductions.

It follows from the above definition,

(i) $\mathbf{a}=a\,\hat{\mathbf{a}}$ where $a=|\mathbf{a}|$ and $\hat{\mathbf{a}}$ is the unit vector in the direction of $\mathbf{a}$.

(ii) $-\mathbf{a}=(-1)\,\mathbf{a}$ *i.e.*, negative of the vector $\mathbf{a}$ is the vector $\mathbf{a}$ multiplied by the scalar -1.

(iii) $0\mathbf{a}=\mathbf{0}$.

(iv) $m\mathbf{a}=\mathbf{a}m=\hat{\mathbf{a}}\,(am)=(am)\,\hat{\mathbf{a}}=m\,(a\hat{\mathbf{a}})$. This is the *Commutative law* of a scalar multiple of a vector.

(v) $m\,(n\mathbf{a})=n\,(m\mathbf{a})=(mn)\,\mathbf{a}$, where m and n are any two scalars. This is the *Associative law* for the multiplication of a vector by scalars.

All these deductions directly follow from the definition and can be easily think through. We now prove the *Distributive law* which states :

$$(m+n)\mathbf{a}=m\mathbf{a}+n\mathbf{a}\ ;\ m\,(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}.$$

The first part again follows from definition. For the second part, we consider two cases :

Case 1. Suppose m is positive.

We refer to Fig. 1.10.

Let $\quad \overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b}$ so that $\overrightarrow{OB} = \mathbf{a} + \mathbf{b}$. $\quad \cdots$ (1)

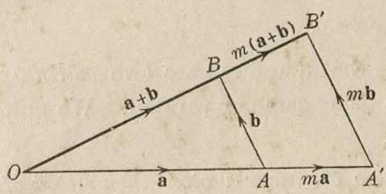


Fig. 1.10.
Distributive law (m positive)

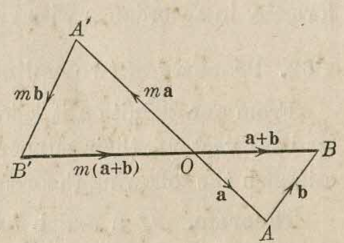


Fig. 1.11.
Distributive law (m negative)

Take a point A' on OA such that $OA' : OA = m : 1$.

If $0 < m < 1$ then A' will be a point lying between O and A, but if $1 < m < \infty$ then A' will lie outside OA on the right of A (see the figure).

Through A' draw $A'B'$ parallel to AB meeting OB (produced, when necessary) in B'. Then from similar triangles OAB, $OA'B'$ we have

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{A'B'}{AB} = m \qquad \ldots \quad (2)$$

Now $\overrightarrow{OA'} = m\ \overrightarrow{OA}$ (by construction) $= m\mathbf{a}$ and by using (2),

$$\overrightarrow{A'B'} = m\ \overrightarrow{AB} = m\mathbf{b},\ \overrightarrow{OB'} = m\ \overrightarrow{OB} = m\,(\mathbf{a} + \mathbf{b}).$$

$\therefore$ By triangle law, we get from $\triangle OA'B'$

$$\overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{A'B'}$$

i.e., $\qquad m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$

Case 2. Suppose m is negative.

We refer to Fig. 1.11 and ask the students to prove the law :

$$m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}.$$

Important Observations. We are now in a position to state that as far as addition, subtraction, multiplication by *numbers* (or *scalars*) are concerned, vectors may be treated formally in accordance with rules of ordinary algebra of numbers.

1·62. Property of two collinear vectors.

From our definition it follows that $\mathbf{a}$ and $m\mathbf{a}$ are two *collinear* (or like) vectors, their supports being same or parallel. We now establish the following theorem :

Theorem. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *be two collinear vectors then either of them can be expressed as the product of the other by a suitable scalar, the numerical value of the scalar being the ratio of the lengths of* $\mathbf{a}$ *and* $\mathbf{b}$, *and conversely.*

Proof. We may write $\mathbf{a}=a\,\hat{\mathbf{a}}$ and $\mathbf{b}=b\,\hat{\mathbf{b}}$ where a, b are the lengths and $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ are unit vectors of $\mathbf{a}$ and $\mathbf{b}$. If the two vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear then $\hat{\mathbf{a}}=\hat{\mathbf{b}}$.

Now, $$\mathbf{b}=b\,\hat{\mathbf{b}}=\frac{b}{a}\cdot a\,\hat{\mathbf{b}}=\frac{b}{a}\,a\,\hat{\mathbf{a}}=\frac{b}{a}\,\mathbf{a}=x\,\mathbf{a}$$

where $x=$ (length of $\mathbf{b}$)/(length of $\mathbf{a}$).

Conversely, if $\mathbf{a}\neq\mathbf{0}$, then any vector $\mathbf{b}$, collinear with $\mathbf{a}$ can be expressed as $x\mathbf{a}$ where x is a suitable scalar. This follows from definition of multiplication of a vector by a scalar.

1·7. Position Vector.

In order to specify the position of a point P in space we require to choose an arbitrary point O as origin. The vector directed from O to P then determines the position of P relative to O. We say that the vector $\overrightarrow{OP}$ is the *position vector* of P with reference to the origin O.

It is customary to denote the position vectors of the points $A, B, C, D, E, F, P, Q, R$ etc., by **a, b, c, d, e, f, p, q, r** etc., the origin O being chosen in advance and supposed to be known even if no specific mention be made. Thus the *point* **a** will mean the position vector of the *point* A referred to a chosen origin.

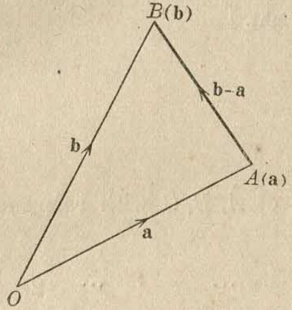


Fig. 1.12—Difference of position vectors

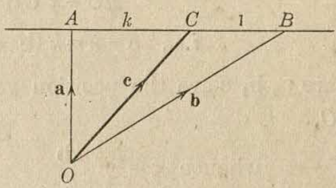


Fig. 1.13—Section ratio

We may express *any vector* $\overrightarrow{AB}$ *as the difference of position vectors of the end points A and B.* Thus, in Fig. 1.12 where O is the origin, we obtain,

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=-\overrightarrow{OA}+\overrightarrow{OB}=-\mathbf{a}+\mathbf{b}=\mathbf{b}-\mathbf{a}.$$

$$=\text{Position vector of } B-\text{Position vector of } A \quad \cdots \quad (1)$$

The relation (1) is very useful. We shall, in future, frequently use $\overrightarrow{CD}=\mathbf{d}-\mathbf{c}$; $\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}$; $\overrightarrow{AH}=\mathbf{h}-\mathbf{a}$; etc.

1·71. Section ratio (or point of division).

Referring to Fig. 1.13 we find that C divides the segment AB in a certain ratio, say in the ratio $k : 1$, so that

$$AC : CB = k : 1 \quad \ldots \quad \ldots \quad (1)$$

Clearly k is positive or negative according as C lies *within* or *outside* the segment AB :

[To the left of A, k varies between 0 and -1 (0 when C coincides with A and -1 when C is at infinite distance to the left of A) ; to the right of B, k varies between $-\infty$ and -1 ($-\infty$ at B and -1 at an infinite distance from B) ; as C describes the line from A to B, k varies from 0 to ∞.

We may remember the variation of k as

$$-1 < k \leq 0\ (A)\ ;\ 0 < k < \infty\ (B)\ ;\ -\infty < k < -1$$

when C describes the entire line from left to right.]

In view of (1) we may write

$$\overrightarrow{AC} = k\ \overrightarrow{CB}$$

$$i.e.,\quad \mathbf{c} - \mathbf{a} = k\,(\mathbf{b} - \mathbf{c}),$$

where **a**, **b**, **c** are the position vectors of A, B, C with reference to O,

$$\text{whence}\quad \mathbf{c} = \frac{\mathbf{a} + k\mathbf{b}}{1+k} \qquad \cdots \qquad \cdots \quad (2)$$

Corollary 1. If $k = m/n$ then $\mathbf{c} = (m\mathbf{b} + n\mathbf{a})/(m+n)$.

e.g., $\frac{1}{5}(2\mathbf{a} + 3\mathbf{b}) = \mathbf{c}$ implies that the point C divides AB in the ratio 3 : 2.

Corollary 2. If $k = 1$, then $\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$; C is then the mid-point of AB.

e.g., the midpoints of BC, CA, AB of a $\triangle ABC$ are $\frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\frac{1}{2}(\mathbf{c} + \mathbf{a})$, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ respectively.

Corollary 3. The result of Cor. 1 may be restated as

$$n\ \overrightarrow{OA} + m\ \overrightarrow{OB} = (m+n)\ \overrightarrow{OC}$$

where C divides AB in the ratio $m : n$. This form of the result will be sometimes useful.

1·72. Two important theorems on Section ratio.

Theorem 1. *The necessary and sufficient condition for three distinct points A, B, C to be collinear is that there should exist three scalars x, y, z, not all zero, such that*

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}\ ;\ x + y + z = 0 \qquad \ldots \quad (1)$$

where **a**, **b**, **c** *are the position vectors of the points* A, B, C *respectively referred to a chosen origin.*

Proof. The condition is necessary.

Suppose A, B, C are collinear. Then C must divide AB in some ratio $m : n$ (positive or negative); also $m/n \neq 0$, ∞ or -1 and so m, n, $m+n$ are not zero. Hence we are justified to write

$$\mathbf{c} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$

$$or, \quad -(m+n)\,\mathbf{c} + m\mathbf{b} + n\mathbf{a} = \mathbf{0} \qquad \dots \quad (2)$$

Put $x = n$, $y = m$, $z = -(n+m)$ then both relations of (1) will follow. Thus if we assume the collinearity of **a**, **b**, **c** then there exist scalars x, y, z, not all zero such that the two relations of (1) hold.

The condition is sufficient.

Now we assume the relations (1) to hold. Since x, y, z are not all zero, we may take one of them not zero (say $z \neq 0$). Then on dividing $x\mathbf{a} + y\mathbf{b} = -z\mathbf{c}$ by $x + y = -z$ we obtain,

$$\frac{x\mathbf{a} + y\mathbf{b}}{x+y} = \mathbf{c}\,;$$

this shows that **c** lies on the join of **a** and **b** and divides the segment in the ratio $y : x$. In other words, the three points **a, b, c** are *termino-collinear*.

Note. The symmetrical relations (1) disclose how each point divides the segment formed by the other two. Thus writing

$$\mathbf{a} = \frac{y\mathbf{b} + z\mathbf{c}}{y+z}\,; \quad \mathbf{b} = \frac{x\mathbf{a} + z\mathbf{c}}{x+z}$$

we conclude that **a** divides the join of **b** and **c** in the ratio $z : y$ and **b** divides the join of **c** and **a** in the ratio $x : z$, the product of the three ratios y/x, z/y, x/z being 1.

Theorem 2. *The necessary and sufficient condition for four points* A, B, C, D, *no three collinear, to be coplanar is that there should exist four scalars,* x, y, z, t, *not all zero, such that*

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = \mathbf{0}\,; \; x + y + z + t = 0 \qquad \cdots \quad (1)$$

where **a**, **b**, **c**, **d** *are the position vectors of A, B, C, D respectively relative a chosen origin O.*

Proof. The condition is necessary.

Suppose A, B, C, D are four points, no three collinear. If they are coplanar then we can always select two of them whose join would divide the join of the other two at some point—for definiteness, say AB divides CD at P. Suppose further that P divides AB in the ratio $m : n$ and CD in the ratio $p : q$. Then, using the notations for position vectors, we may write

$$\mathbf{p}=\frac{n\mathbf{a}+m\mathbf{b}}{n+m}; \quad \mathbf{p}=\frac{q\mathbf{c}+p\mathbf{d}}{q+p} \qquad \cdots \quad (2)$$

where m/n, p/q are neither 0 nor -1. Hence, from (2), we get by equating the two expressions for **p**,

$$\frac{n}{n+m}\mathbf{a}+\frac{m}{n+m}\mathbf{b}-\frac{q}{q+p}\mathbf{c}-\frac{p}{q+p}\mathbf{d}=\mathbf{0} \qquad \cdots \quad (3)$$

Taking $x=n/(n+m)$, $y=m/(n+m)$, $z=-q/(q+p)$, $t=-p/(q+p)$, we get the relations (1). Thus for four coplanar points **a**, **b**, **c**, **d** there exist scalars, not all zero, such that the relations (1) are true.

The condition is sufficient.

Here we assume the relations (1) and observe that one, at least, of $x+y$, $z+t$, $x+t$ is not zero, for if they were all zero, then we would have $x=y=z=t=0$. Thus we may assume $z+t \neq 0$. Then $z+t=-(x+y)$ and on dividing $x\mathbf{a}+y\mathbf{b}=-(z\mathbf{c}+t\mathbf{d})$ by $(x+y)$ and $-(z+t)$ we get

$$\frac{x\mathbf{a}+y\mathbf{b}}{x+y}=\frac{z\mathbf{c}+t\mathbf{d}}{z+t}=\mathbf{p} \text{ (say)}.$$

This proves that there exists a point P which divides AB in the ratio $y : x$ and CD in the ratio $t : z$. In other words, AB intersects CD at P *i.e.*, four points A, B, C, D are coplanar. We sometimes call them *termino-coplanar*.

Note that even if $z+t=0$ (and hence $x+y=0$) we have

$$x(\mathbf{a}-\mathbf{b})=-t(\mathbf{c}-\mathbf{d})=t(\mathbf{d}-\mathbf{c})$$

i.e., $x\,\overrightarrow{BA}=t\,\overrightarrow{CD}$,

i.e., AB and CD are parallel and four points are again coplanar.

Note. If four points A, B, C, D form a plane quadrilateral then there exist relations of the form

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+t\mathbf{d}=\mathbf{0}\ ;\ x+y+z+t=0$$

whence, if $x+z \neq 0$, then

$$\frac{x\mathbf{a}+z\mathbf{c}}{x+z}=\frac{y\mathbf{b}+t\mathbf{d}}{y+t}=\mathbf{q}\ (\text{say})\ ;$$

this discloses that Q divides AC in the ratio $z:x$ and BD in the ratio $t:y$. If AD and BC meet at R, how does R divide these segments ?

Examples. I(A)

1. Find the position vector of C if it divides AB in the ratio (i) $2:3$; (ii) $2:-3$.

2. Find the point which divides the join of $\mathbf{p}$ and $\mathbf{q}$ internally and externally in the ratio $3:4$.

3. CD is trisected at P. Find the point of trisection.

4. ABC is a triangle. D divides BC in the ratio $l:m$; G divides AD in the ratio $l+m:n$. Find the position vectors of D and G.

5. Suppose in Example 4, D is the midpoint of BC and G divides AD in the ratio $2:1$. Find the position vectors of D and G. (G is then called the *centroid* of $\triangle ABC$).

6. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third and half of it.

7. If D, E, F be the midpoints of the sides BC, CA, AB of a triangle ABC, prove that

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\mathbf{0}$$

and further, prove that the medians of a triangle are concurrent.

8. Show that if $\overrightarrow{AB}=\overrightarrow{DC}$ then the figure $ABCD$ is a parallelogram and prove that the diagonals bisect each other.

9. Show, conversely that if the diagonals of the quadrilateral $ABCD$ bisect each other then the figure is a parallelogram.

10. $ABCD$ is a parallelogram. P, Q are the midpoints of the sides AB and CD respectively; show that DP and BQ trisect AC and are trisected by AC.

11. If $ABCD$ be a quadrilateral (plane or skew) show that the midpoints of its sides are the vertices of a parallelogram and the joins of the midpoints of the opposite edges bisect each other.

12. We know that the internal bisector of any angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle. Assuming this, show that the internal bisectors of angles of a triangle are concurrent.

13. P divides AB in the ratio $m : n$; Q divides AB in the ratio $-m : n$. Then P and Q are called harmonic conjugates of A, B. Show that A, B are also harmonic conjugates of P, Q.

14. The parallel sides AB, CD of a trapezoid are in the ratio $x : y$. Prove that AC, BD divide at F (say) in the same ratio $x : y$ and that AD, BC divide at E (say) in the ratio $-x : y$. Prove that EF produced bisects AB. Find also the ratio in which AB divides EF.

15. The points L, M, N divide the sides BC, CA, AB of a $\triangle ABC$ in the ratios $1 : 4$, $3 : 2$ and $3 : 7$ respectively. Prove that $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$ where K divides AB in the ratio $1 : 3$.

16. The straight line joining the midpoints of two non-parallel sides of a trapezium is parallel to the parallel sides and half their sum.

17. $ABCD$ is a plane quadrilateral. The position vectors of its vertices satisfy the relation $2\mathbf{a}+3\mathbf{b}-\mathbf{c}-4\mathbf{d}=\mathbf{0}$. Find the

ratios in which P divides AB and CD, P being the point of intersection of AB and CD.

18. In the previous example, suppose AC and BD meet at Q; AD and BC meet at R. Find the ratios in which Q divides AC and BD. Also find the ratios in which R divides AD and BC.

19. $ABCD$ is a quadrilateral. The diagonals AC and BD meet at P. The sides AB, CD meet at Q and BC, DA meet at R. P divides AC in the ratio $3:2$, BD in the ratio $1:2$. Find the ratios in which Q divides AB and CD, R divides BC and DA.

20. If $3\mathbf{a}-2\mathbf{b}+\mathbf{c}-2\mathbf{d}=\mathbf{0}$, are the points A, B, C, D coplanar? Find the point P where AC and BD meet. In what ratio does P divide AC and BD?

Note. In what follows, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ denote any three non-coplanar vectors.

21. Show that the three points

$$\mathbf{a}-2\mathbf{b}+3\mathbf{c}, \quad -2\mathbf{a}+3\mathbf{b}+2\mathbf{c}, \quad -8\mathbf{a}+13\mathbf{b}$$

are collinear.

22. Verify whether the three points

$$\mathbf{a}-2\mathbf{b}+3\mathbf{c}, \quad 2\mathbf{a}+3\mathbf{b}-4\mathbf{c}, \quad -7\mathbf{b}+10\mathbf{c}$$

are collinear or not?

23. Show that the four points in the following two cases are coplanar:

(i) $6\mathbf{a}-4\mathbf{b}+10\mathbf{c}$, $-5\mathbf{a}+3\mathbf{b}-10\mathbf{c}$, $4\mathbf{a}-6\mathbf{b}-10\mathbf{c}$, $2\mathbf{b}+10\mathbf{c}$.

(ii) $-\mathbf{a}+4\mathbf{b}-3\mathbf{c}$, $3\mathbf{a}+2\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+8\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+2\mathbf{b}+\mathbf{c}$.

Note. Any point can be expressed as a linear combination of three non-coplanar vectors—this fact has been assumed tacitly in Ex. 21, 22, 23. The justification will be found in art. 1·93.

Hints and Answers.

1. $\frac{1}{5}(3\mathbf{a}+2\mathbf{b})$; $3\mathbf{a}-2\mathbf{b}$. 2. $\frac{1}{7}(4\mathbf{p}+3\mathbf{q})$; $4\mathbf{p}-3\mathbf{q}$.

3. $\frac{1}{3}(\mathbf{c}+2\mathbf{d})$. 4. $\mathbf{d}=\dfrac{l\mathbf{c}+m\mathbf{b}}{l+m}$; $\mathbf{g}=\dfrac{(l+m)\mathbf{d}+n\mathbf{a}}{l+m+n}=\dfrac{l\mathbf{c}+m\mathbf{b}+n\mathbf{a}}{l+m+n}$.

5. $\frac{1}{2}(\mathbf{b}+\mathbf{c})$, $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. 6. With the notations of Ex. 7, $\mathbf{d}=\frac{1}{2}(\mathbf{b}+\mathbf{c})$, $\mathbf{e}=\frac{1}{2}(\mathbf{c}+\mathbf{a})$, $\mathbf{f}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$. Hence $\overrightarrow{DE}=\mathbf{e}-\mathbf{d}=\frac{1}{2}(\mathbf{a}-\mathbf{b})=\frac{1}{2}\overrightarrow{BA}$; hence etc.

7. (i) $\overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}$; similarly obtain $\overrightarrow{BE}$, $\overrightarrow{CF}$ and then add.

(ii) On AD take a point G such that $AG : GD=2 : 1$ then $\mathbf{g}=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Take a point G' on BE such that $BG' : G'E=2 : 1$; then again $\mathbf{g}'=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Similarly for a point G'' on CF such that $CG'' : G''F=2 : 1$ we obtain, $\mathbf{g}''=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Thus there exist three points $\mathbf{g}, \mathbf{g}', \mathbf{g}''$, one on each median which have the same position vector $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. That is, the three medians meet at this point, called *centroid of the triangle*. Further it proves that centroid divides each median in the ratio 2 : 1.

8. (i) $\overrightarrow{AB}=\overrightarrow{DC}$ gives $\mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$; or $\mathbf{d}-\mathbf{a}=\mathbf{c}-\mathbf{b}$ *i.e.*, $\overrightarrow{AD}=\overrightarrow{BC}$. Hence $ABCD$ is a parallelogram.

(ii) Again from $\mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$ we construct $\frac{1}{2}(\mathbf{a}+\mathbf{c})=\frac{1}{2}(\mathbf{b}+\mathbf{d})$ which will prove the second part.

9. Assume $\frac{1}{2}(\mathbf{a}+\mathbf{c})=\frac{1}{2}(\mathbf{b}+\mathbf{d})$; now obtain $\mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$ *i.e.*, $\overrightarrow{AB}=\overrightarrow{DC}$; hence etc.

10. $\mathbf{p}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$, $\mathbf{q}=\frac{1}{2}(\mathbf{c}+\mathbf{d})$. Since $ABCD$ is a parallelogram $\mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$ whence $\mathbf{b}=\mathbf{a}+\mathbf{c}-\mathbf{d}$; but $2\mathbf{p}=\mathbf{a}+\mathbf{b}$; it follows $\mathbf{a}+\mathbf{c}-\mathbf{d}=2\mathbf{p}-\mathbf{a}$, or $\frac{1}{3}(2\mathbf{p}+\mathbf{d})=\frac{1}{3}(2\mathbf{a}+\mathbf{c})$; hence etc.

11. Obtain the midpoints and show that they form a parallelogram whose centre is $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$.

12. We denote the lengths of sides by a, b, c. According to the problem if the internal bisector of $\angle BAC$ intersects the opposite side BC at D, then $BD : DC=c : b$ and hence $\mathbf{d}=\dfrac{c\mathbf{c}+b\mathbf{b}}{c+b}$. Take a point I on AD such that $AI : ID=b+c : a$. Then position vector of $I=\dfrac{a\mathbf{b}+b\mathbf{b}+c\mathbf{c}}{a+b+c}$. The symmetry gives the result.

13. $\mathbf{p}=\dfrac{n\mathbf{a}+m\mathbf{b}}{n+m}$, $\mathbf{q}=\dfrac{n\mathbf{a}-m\mathbf{b}}{n-m}$; hence deduce $2n\mathbf{a}=(n+m)\mathbf{p}+(n-m)\mathbf{q}$; $2m\mathbf{b}=(n+m)\mathbf{p}-(n-m)\mathbf{q}$. Thus A, B divide PQ in the ratios $\pm(n-m):(n+m)$.

14. $\overrightarrow{AB}=(x/y)\,\overrightarrow{DC}$ gives $y(\mathbf{b}-\mathbf{a})=x(\mathbf{c}-\mathbf{d})$ whence it follows $\mathbf{f}=\dfrac{y\mathbf{b}+x\mathbf{d}}{y+x}=\dfrac{x\mathbf{c}+y\mathbf{a}}{x+y}$ and $\mathbf{e}=\dfrac{y\mathbf{a}-x\mathbf{d}}{y-x}=\dfrac{y\mathbf{b}-x\mathbf{c}}{y-x}$. Now consider $(x\mathbf{c}+y\mathbf{a})=(x+y)\mathbf{f}$;

$(-x\mathbf{c}+y\mathbf{b})=(y-x)\mathbf{e}$. Eliminate $\mathbf{c}$ and deduce $\mathbf{k}=\frac{y\mathbf{a}+y\mathbf{b}}{2y}=\frac{(x+y)\mathbf{f}+(y-x)\mathbf{e}}{(x+y)+(y-x)}$. Thus $\mathbf{k}$ divides AB in the ratio 1 : 1 and EF in the ratio $y+x : y-x$; hence etc.

17. Writing the given relation as $\frac{2\mathbf{a}+3\mathbf{b}}{2+3}=\frac{\mathbf{c}+4\mathbf{d}}{1+4}$ see that P divides AB in the ratio 3 : 2 and CD in the ratio 4 : 1.

18. $-\frac{1}{2}, -\frac{4}{3}$; $-\frac{2}{1}, -\frac{1}{3}$.

19. $\mathbf{p}=\frac{1}{5}(2\mathbf{a}+3\mathbf{c})=\frac{1}{3}(2\mathbf{b}+\mathbf{d})$ whence, $6\mathbf{a}-10\mathbf{b}+9\mathbf{c}-5\mathbf{d}=\mathbf{0}$. Now proceed as in Ex. 17.

20. 1 : 3 and 1 : 1.

21. Denote three points by $\mathbf{p}, \mathbf{q}, \mathbf{r}$. Now see that $2\mathbf{p}+(-3)\mathbf{q}+\mathbf{r}=\mathbf{0}$ where the sum of the coeff. is equal to zero ; hence they are collinear.

22. They are collinear.

23. (i) Verify $3\mathbf{p}+2\mathbf{q}-2\mathbf{r}-3\mathbf{s}=\mathbf{0}$ where sum of coeff. $=0$; $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ being the four given points.

1·8. Linear combination.

DEFINITION. If a vector $\mathbf{r}$ can be expressed as

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\cdots,$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}\ldots$ are a finite number of vectors and $x, y, z, \ldots$ are scalars then $\mathbf{r}$ is called a *linear combination* of the set of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

Illustrations. (i) If $\mathbf{a}$ and $\mathbf{b}$ are collinear then we know $\mathbf{b}=x\mathbf{a}$. Thus $\mathbf{b}$ is a linear combination of $\mathbf{a}$.

(ii) Any vector $\mathbf{c}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be written as $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$ (art 1.9). Thus $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

(iii) Any vector $\mathbf{d}$ can be expressed as $\mathbf{d}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$ where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any three non-coplanar vectors (art 1.93). Thus $\mathbf{d}$ is a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

1·81. Linearly dependent and independent system of vectors.

DEFINITION. If the scalars $x, y, z, \ldots$, not all zero, exist such that $$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\cdots=\mathbf{0}$$

then the set $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ of vectors is said to form a *linearly dependent system* of vectors ; otherwise they form a *linearly independent system.* Thus if $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ form a linearly independent system then

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \cdots = \mathbf{0}$$

will imply $$x = y = z = \cdots = 0.$$

Illustrations. (i) If $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors then we may obtain $\mathbf{b} - x\mathbf{a} = \mathbf{0}$, where x is some non-zero scalar. Thus $\mathbf{a}$ and $\mathbf{b}$ are *linearly dependent.*

(ii) For three coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we may write $\mathbf{c} - x\mathbf{a} - y\mathbf{b} = \mathbf{0}$ *i.e.*, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are *linearly dependent.*

(iii) For any four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ we have always a relation of the form $\mathbf{d} - x\mathbf{a} - y\mathbf{b} - z\mathbf{c} = \mathbf{0}$. Hence four vectors are always *linearly dependent.*

(iv) Two *non-collinear vectors* $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent,* because the relation $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$ would imply $x = 0$, $y = 0$. For, if $x \neq 0$, then $\mathbf{a} = (-y/x)\mathbf{b}$ which would imply the collinearity of $\mathbf{a}$ and $\mathbf{b}$, a contradiction to our assumption. Hence $x = 0$, similarly $y = 0$.

(v) Three *non-coplanar vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are *linearly independent,* for the relation $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$ would imply $x = 0$, $y = 0$, $z = 0$. If $x \neq 0$ then $\mathbf{a} = (-y/x)\mathbf{b} + (-z/x)\mathbf{c}$ which would imply the coplanarity of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, a contradiction to our assumption. Hence $x = 0$. Similarly $y = 0$, $z = 0$.

(vi) Any four or *more vectors* always form a *linearly dependent system.*

1·9. Resolution of a vector : Coplanar vectors.

Theorem. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are two non-collinear vectors. Any vector* $\mathbf{c}$, *coplanar with* $\mathbf{a}$ *and* $\mathbf{b}$ *can be expressed as a linear combination of* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof. Take O as origin and suppose (Fig. 1.14)

$$\overrightarrow{OX}=\mathbf{a},\ \overrightarrow{OY}=\mathbf{b},\ \overrightarrow{OP}=\mathbf{c}.$$

The lines OX, OY and OP are supposed to be coplanar. With OP as diagonal and adjacent sides along **a** and **b** we construct a parallelogram $OAPB$. Then

$$\overrightarrow{OA}=x\mathbf{a},\ \overrightarrow{OB}=y\mathbf{b}$$

(x, y are suitably chosen scalars).

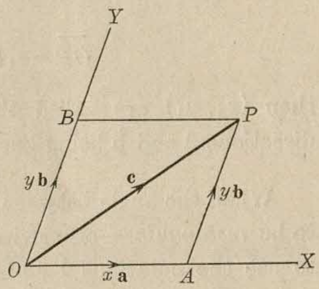


Fig. 1.14. Resolution of a vector in a plane

whence,

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\overrightarrow{OB}=x\mathbf{a}+y\mathbf{b}$$

i.e., $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$.

Uniqueness. This resolution of **c** is unique since only one parallelogram with OP as diagonal and adjacent sides along **a** and **b** can be constructed.

Alternative Proof for uniqueness. If possible, let $\mathbf{c}=x'\mathbf{a}+y'\mathbf{b}$. Then $x\mathbf{a}+y\mathbf{b}=x'\mathbf{a}+y'\mathbf{b}$ *i.e.*, $(x-x')\mathbf{a}=(y'-y)\mathbf{b}$ which shows that **a** and **b** are collinear, *a contradiction to our assumption*, unless $x-x'=0$ and $y-y'=0$ [see art. 1.81 Illus. (iv)].

Corollary. If **a**, **b**, **c** are coplanar, they form a linearly dependent system ; they form a linearly independent system if they are non-coplanar.

1·91. Coördinates of a two-dimensional vector.

Take O as origin and suppose

$$\overrightarrow{OX}=\mathbf{a},\ \overrightarrow{OY}=\mathbf{b}.$$

We have noted that the position vector of P with reference to O is given by

$$\overrightarrow{OP}=x\mathbf{a}+y\mathbf{b}\qquad (x,\ y \text{ are suitable scalars}).$$

We call $x\mathbf{a}$, $y\mathbf{b}$ the *components* of $\overrightarrow{OP}$ along $\mathbf{a}$ and $\mathbf{b}$ respectively ; the scalar coefficients x, y are called *Affine coördinates* of P. In particular, if $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors along $\mathbf{a}$ and $\mathbf{b}$ and

$$\overrightarrow{OP} = x_1\hat{\mathbf{a}} + y_1\hat{\mathbf{b}},$$

then (x_1, y_1) are called the *Cartesian coördinates* of P, the directions $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ being known as the *axes of coördinates.*

When the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is $90°$ we call the coördinates to be *rectangular*—otherwise, they are *oblique.* It is customary to use the notations $\mathbf{i}$ and $\mathbf{j}$ for the unit vectors along the two axes OX, OY in rectangular cartesian system of coördinates (Fig. 1.15). Thus if

$$\overrightarrow{OP} = x_1\,\mathbf{i} + y_1\,\mathbf{j},$$

we say that the *rectangular cartesian coördinates* of P are (x_1, y_1). In fact, we may identify the vector $\overrightarrow{OP}$ with the aggregate of two numbers, *taken in order* and write

$$\overrightarrow{OP} = (x_1, y_1) \;;$$

(x_1, y_1) are also known as *coördinates of the two dimensional vector* $\overrightarrow{OP}$. According to this notation,

$$\mathbf{i} = (1, 0) \;;\; \mathbf{j} = (0, 1) \;;\; \mathbf{O} = (0, 0).$$

From the definition of a two-dimensional vector as an ordered pair of numbers we have the following operations defined by

Addition : $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Subtraction : $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$.

Scalar multiplication : $k(x_1, y_1) = (kx_1, ky_1)$.

Equality : $(x_1, y_1) = (x_2, y_2)$, if and only if $x_1 = x_2$, $y_1 = y_2$.

Section Ratio. If $\mathbf{c}=(x, y)$ divides the join of $\mathbf{a}=(x_1, y_1)$ and $\mathbf{b}=(x_2, y_2)$ in the ratio $k:1$ then

$$(x, y)=\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}\right); \left(\because \ \mathbf{c}=\frac{\mathbf{a}+k\mathbf{b}}{1+k}\right)$$

giving two familiar scalar equations

$$x=\frac{x_1+kx_2}{1+k}, \quad y=\frac{y_1+ky_2}{1+k}$$

which are the coördinates of a point dividing the segment whose ends are (x_1, y_1) and (x_2, y_2) in the ratio $k:1$.

1·92. Geometric Interpretations of Algebra of ordered numbers.

We next consider the converse problem. Suppose we define a vector $\mathbf{c}$ as an ordered pair of numbers, say

$$\mathbf{c}=(x_1, y_1).$$

What geometric significance can be associated with it? We agree to locate a point P whose coördinates are (x_1, y_1) referred to a pair of axes (usually, rectangular); see Fig. 1.15. We further agree to write $\overrightarrow{OP}=(x_1, y_1)$. Also any other parallel directed segment, say $\overrightarrow{AB}$ or $\overrightarrow{CD}$ whose projections are x_1, y_1 on the two axes may be supposed to correspond the pair (x_1, y_1).

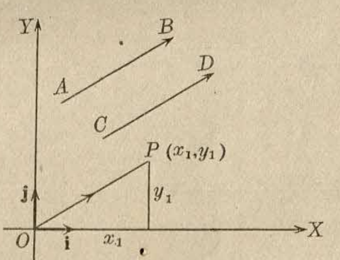


Fig. 1.15. coördinates of a vector
(Two-dimensional)

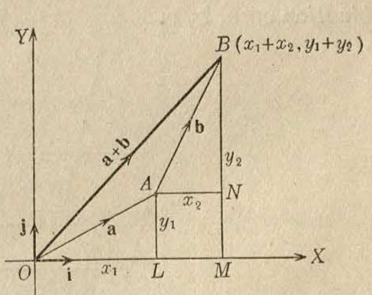


Fig. 1.16.
Sum of two ordered pairs

Geometric Interpretation of :

$$(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2). \quad \dots \quad (1)$$

In Fig. 1.16 we take (x_1, y_1) the coördinates of A with reference to the axes OX, OY and x_2, y_2 as the projections of the segment AB on the axes. Then from simple geometric considerations it follows that the coördinates of B are (x_1+x_2, y_1+y_2) so that

$$\overrightarrow{OB}=(x_1+x_2, y_1+y_2).$$

Hence in vectorial notations, (1) becomes

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB},$$

which gives the familiar triangle law of addition of vectors. We may similarly interpret

$$(x_1, y_1)-(x_2, y_2)=(x_1-x_2, y_1-y_2);$$
$$k(x_1, y_1)=(kx_1, ky_1).$$

Finally, we remark that the algebra of ordered numbers is entirely equivalent to vector algebra (based on the notion of directed segments). Moreover it is free from the concept of Geometry and readily lends itself to generalisations.

1·93. Resolution of a vector in three-dimensional space.

Theorem. *If* **a, b, c** *are any three non-coplanar vectors then any vector* **r** *can be expressed uniquely as a linear combination of* **a, b, c.**

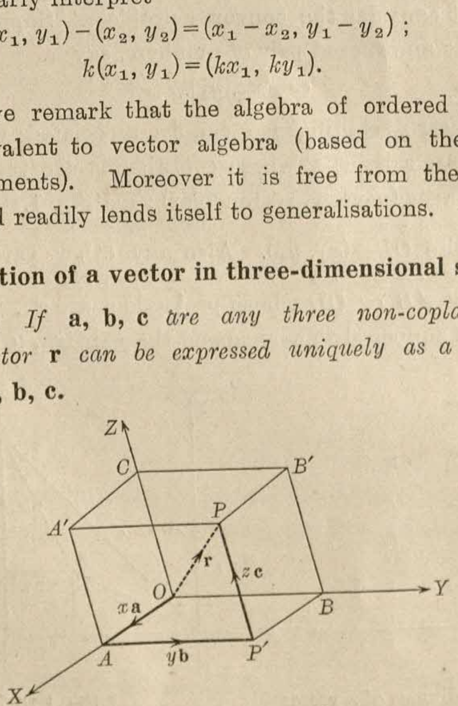


Fig. 1.17—Resolution of a three-dimensional vector.

Proof. Take O as origin (Fig. 1.17).

Let $\overrightarrow{OX}=\mathbf{a}$, $\overrightarrow{OY}=\mathbf{b}$, $\overrightarrow{OZ}=\mathbf{c}$, $\overrightarrow{OP}=\mathbf{r}$.

With OP as diagonal construct a parallelopiped with edges along **a, b, c.**

Now $\overrightarrow{OA}=x\mathbf{a}$, $\overrightarrow{OB}=y\mathbf{b}$, $\overrightarrow{OC}=z\mathbf{c}$; where x, y, z are suitably chosen scalars. Then, $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP'}+\overrightarrow{P'P}$ so that

$$\mathbf{r}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}.$$

This shows that $\mathbf{r}$ is a linear combination of **a, b, c.**

Uniqueness. This resolution is unique since only one parallelopiped having OP as diagonal and edges along **a, b, c** can be constructed.

Alternative Proof of uniqueness. If possible, let $\mathbf{r}=x'\mathbf{a}+y\mathbf{b}'+z'\mathbf{c}$, then from

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=x'\mathbf{a}+y'\mathbf{b}+z'\mathbf{c}$$

it follows, $\quad (x-x')\,\mathbf{a}+(y-y')\,\mathbf{b}+(z-z')\,\mathbf{c}=\mathbf{0}$

which shows that they are coplanar, a *contradiction to our assumption* unless $x-x'=0$, $y-y'=0$, $z-z'=0$ [See art. 1·81, Illus. (v)]

Corollary. Any four vectors are linearly dependent.

1·94. Coördinates of a three-dimensional vector.

Take O as origin and suppose (Fig. 1.17)

$$\overrightarrow{OX}=\mathbf{a},\quad \overrightarrow{OY}=\mathbf{b},\quad \overrightarrow{OZ}=\mathbf{c}.$$

We have noted that the position vector of P with reference to O as origin is given by

$$\overrightarrow{OP}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c} \quad (x, y, z \text{ are suitable scalars}).$$

We call $x\mathbf{a}$, $y\mathbf{b}$, $z\mathbf{c}$ the *components* of $\overrightarrow{OP}$ along **a, b, c** respectively ; the scalar coefficients x, y, z are called *Affine coördinates* of P.

In particular, if $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors along **a, b, c** and if

$$\overrightarrow{OP} = x_1\hat{\mathbf{a}} + y_1\hat{\mathbf{b}} + z_1\hat{\mathbf{c}}$$

then (x_1, y_1, z_1) are called the *cartesian coördinates* of P with reference to the directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ as *axes of coördinates*. If the three directions are mutually perpendicular (Fig. 1.18) then the coördinates are rectangular—otherwise *oblique*.

We shall always consider rectangular cartesian coördinates unless otherwise specifically mentioned. It is customary to use the notations **i, j, k** for the unit vectors along three such axes of coördinates OX, OY, OZ.

Thus if $\overrightarrow{OP} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ then (x_1, y_1, z_1) are called the *rectangular cartesian coördinates* of P.

In fact we may identify the vector $\overrightarrow{OP}$ with the aggregate of three numbers, taken in a determined order, and write

$$\overrightarrow{OP} = (x_1, y_1, z_1);$$

(x_1, y_1, z_1) are also called the *coördinates of the three-dimensional vector* $\overrightarrow{OP}$. The fundamental vectors are then given by

$$\mathbf{i} = (1, 0, 0); \ \mathbf{j} = (0, 1, 0); \ \mathbf{k} = (0, 0, 1); \ \mathbf{0} = (0, 0, 0).$$

Exactly as in art. 1·91 we may define the following operations where vectors are defined by number-triples :

Addition. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$.

Subtraction. $(x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$.

Equality. $(x_1, y_1, z_1) = (x_2, y_2, z_2)$, if and only if

$$x_1 = x_2, y_1 = y_2, z_1 = z_2.$$

Scalar Multiplication. $k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$.

Section Ratio. If $\mathbf{c} = (x, y, z)$ divides the join of $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ in the ratio $k : 1$, then

$$(x, y, z) = \left(\frac{x_1 + kx_2}{1+k}, \frac{y_1 + ky_2}{1+k}, \frac{z_1 + kz_2}{1+k}\right).$$

Convention of Signs. We shall always consider the right-handed system of axes. OX, OY, OZ form a *right-handed system of axes* or *a dextral system* if a right-handed rotation of 90° about OZ carries $+OX$ into $+OY$. Such a rotation viewed from $+OZ$ will be counter-clockwise (Fig. 1.18). This system will correspond to the direction of translation of a right-handed screw.

Thus a right-handed screw revolved in a nut from OX to OY will move in the direction OZ.

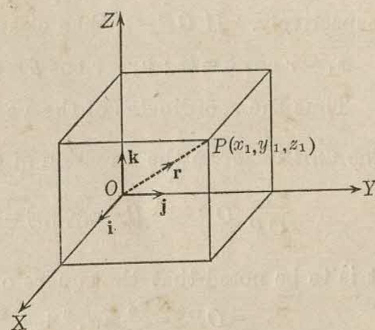


Fig. 1.18. Coördinates of vector (three-dimensional)

1·95. Geometric Interpretations of Algebra of number-triples.

Suppose a vector $\mathbf{r}$ is defined as an ordered number-triples, say

$$\mathbf{r}=(x_1, y_1, z_1).$$

What geometric significance can be associated with it? We agree to associate a point P with coördinates (x_1, y_1, z_1) referred to a system of axes (usually, rectangular); see Fig. 1.18. We further agree to write

$$\overrightarrow{OP}=(x_1, y_1, z_1).$$

Any other segment (not shown in the figure) having projections x_1, y_1, z_1 on three axes will also correspond to the number triple (x_1, y_1, z_1).

Different algebraic operations with number triples can be interpreted as in art. 1·92.

Important Observations. If α, β, γ are the angles which the line OP makes with OX, OY, OZ respectively (the system of

axes being rectangular) then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of $\overrightarrow{OP}$. They are frequently denoted by l, m, n respectively. If $OP=r$, then clearly

$$x_1=r\cos\alpha=lr\ ;\ y_1=r\cos\beta=mr\ ;\ z_1=r\cos\gamma=nr. \quad \cdots \quad (1)$$

Thus the coördinates of the vector are $(rl, rm, rn)=r(l, m, n)$.

The *unit vector* in the direction of $\overrightarrow{OP}$ is

$$\frac{1}{OP}\overrightarrow{OP}=\frac{1}{r}(lr, mr, nr)=(\cos\alpha, \cos\beta, \cos\gamma).$$

It is to be noted that the square of modulus of $\overrightarrow{OP}$

$$=OP^2=r^2=x_1^{\,2}+y_1^{\,2}+z_1^{\,2}$$

whence it also follows from (1),

$$l^2+m^2+n^2=1,$$

an important relation of direction cosines of a line.

Examples. I(B)

1. If $\mathbf{a}=(1, -2, 3)$, $\mathbf{b}=(-2, 3, -4)$, find $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $3\mathbf{a}+2\mathbf{b}$, $3\mathbf{a}-2\mathbf{b}$.

2. Prove that $\mathbf{a}=\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+6\mathbf{j}+10\mathbf{k}$ form a linearly dependent system.

3. If $\mathbf{a}=(1, -2, 3)$, $\mathbf{b}=(-2, 3, -4)$, $\mathbf{c}=(0, -1, 2)$, show that they form a linearly dependent system.

4. If $\mathbf{a}=(1, -1, 1)$, $\mathbf{b}=(-1, 1, 1)$, $\mathbf{c}=(1, 1, 1)$ and $\mathbf{d}=(2, -3, 4)$, show that $7\mathbf{a}+2\mathbf{b}-\mathbf{c}-2\mathbf{d}=\mathbf{0}$. Are the four points coplanar ?

5. *ABC* is a triangle ; *D*, *E*, *F* are the mid-points of the sides *BC*, *CA*, *AB* respectively. Express $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ as linear combinations of $\mathbf{a}$ and $\mathbf{b}$ where $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{AC}=\mathbf{b}$. Deduce that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\mathbf{0}$.

6. $OABC$ is a tetrahedron ; $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$. Express $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ in terms of **a**, **b** and **c**.

7. $ABCD$ is a parallelogram whose diagonals are AC and BD. Prove that $\overrightarrow{AC}+\overrightarrow{BD}=2\overrightarrow{BC}$; $\overrightarrow{AC}-\overrightarrow{BD}=2\overrightarrow{AB}$.

8. $ABCDEF$ is a regular hexagon ; $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$. Express the remaining sides and diagonals in terms of **a** and **b**.

9. Find a linear relation between the system of vectors $6\mathbf{a}-4\mathbf{b}+10\mathbf{c}$, $-6\mathbf{a}+4\mathbf{b}-10\mathbf{c}$, $4\mathbf{a}-6\mathbf{b}-10\mathbf{c}$, $-2\mathbf{a}+3\mathbf{b}+5\mathbf{c}$.

10. Find the lengths of sides of the triangle ABC whose vertices have the position vectors $A(3, 4, 5)$; $B(4, 3, 2)$; $C(3, 6, -3)$. Find the direction cosines of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

11. Three points whose position vectors are

$$A(2, 4, -1) ;\ B(4, 5, 1) ;\ C(3, 6, -3)$$

form a triangle ABC. Find the lengths of the sides BC, CA, AB. Hence prove that the triangle is an *isosceles right* triangle. Find also the direction cosines of $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$.

12. Similar problem as No 11 with

$$A(0, 7, 10) ;\ B(-1, 6, 6) ;\ C(-4, 9, 6).$$

13. Show that the three points whose position vectors are

$$A(-2, 3, 5) ;\ B(1, 2, 3) ;\ C(7, 0, -1)$$

lie on a line.

14. $A(3, 2, 0)$; $B(5, 3, 2)$, $C(-9, 6, -3)$ form a triangle ABC. P, Q divide the side BC in the ratio $\pm 3 : 2$. Find the position vectors of P, Q. Find also the direction cosines of $\overrightarrow{AP}$ and $\overrightarrow{AQ}$.

15. In the previous example, suppose AD, the bisector of the angle BAQ, meets BC at D. Find the coördinates of D.

16. Two vectors $\mathbf{a}=(a_1, a_2, a_3)$, $\mathbf{b}=(b_1, b_2, b_3)$ are collinear. Show that $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$. Use this result to show

that the points whose position vectors are $A(2, 5, -4)$, $B(1, 4, -3)$, $C(4, 7, -6)$ are collinear. In what ratio does B divide AC ? Take a point D on AC so that B and D are harmonic conjugates to A, C. Find the position vector of D.

17. Similar problem as No. 16 with

$$A(5, 4, 2)\ ;\ B(6,\ 2,\ -1),\ C(8,\ -2,\ -7).$$

18. Show that the four points whose position vectors are

$$A(4, 8, 12)\ ;\ B(2, 4, 6)\ ;\ C(3, 5, 4)\ ;\ D(5, 8, 5)$$

are coplanar.

19. Show that the four points

$$A(6, -7, 0)\ ;\ B(16, -19, -4)\ ;\ C(0, 3, -6)\ ;\ D(2, 5, 10)$$

are such that AB and CD intersect at the point $P(1, -1, 2)$.

20. If $\mathbf{a}=(a_1, a_2, a_3)$ and $\mathbf{b}=(b_1, b_2, b_3)$ be the position vectors of A, B respectively find the point C on AB (produced) such that $AC=3AB$ and a point D in BA (produced) such that $BD=2BA$.

21. Show that the centroid of a triangle whose vertices are (a_r, b_r, c_r) ; $r=1, 2, 3$ is

$$[\tfrac{1}{3}(a_1+a_2+a_3),\ \tfrac{1}{3}(b_1+b_2+b_3),\ \tfrac{1}{3}(c_1+c_2+c_3)].$$

22. The position of P is given by the vector $3\mathbf{i}+12\mathbf{j}+4\mathbf{k}$. Find the direction cosines of $\overrightarrow{OP}$.

Hints and Answers

1. $\mathbf{a}+\mathbf{b}=(1-2,\ -2+3,\ 3-4)=(-1,\ 1,\ -1)$; $\mathbf{a}-\mathbf{b}=(3,\ -5,\ 7)$; $3\mathbf{a}+2\mathbf{b}=3(1,\ -2,\ 3)+2(-2,\ 3,\ -4)=(3,\ -6,\ 9)+(-4,\ 6,\ -8)=(-1,\ 0,\ 1)$; $3\mathbf{a}-2\mathbf{b}=(7,\ -12,\ 17)$.

2. $2\mathbf{a}-\mathbf{b}=0$; hence etc. 3. $2\mathbf{a}+\mathbf{b}-\mathbf{c}=0$. 4. Not coplanar.

5. $\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AB}+\tfrac{1}{2}(\overrightarrow{BA}+\overrightarrow{AC})=\mathbf{a}-\tfrac{1}{2}\mathbf{a}+\tfrac{1}{2}\mathbf{b}=\tfrac{1}{2}(\mathbf{a}+\mathbf{b})$.

6. $\mathbf{c}-\mathbf{b}$, $\mathbf{a}-\mathbf{c}$, $\mathbf{b}-\mathbf{a}$. 8. Sides taken in order: $\mathbf{b}-\mathbf{a}$; $-\mathbf{a}$; $-\mathbf{b}$; $\mathbf{a}-\mathbf{b}$; Diagonals : $\mathbf{a}+\mathbf{b}$; $2\mathbf{b}$; $2\mathbf{b}-\mathbf{a}$; $2\mathbf{b}-\mathbf{a}$; $2\mathbf{b}-2\mathbf{a}$; $\mathbf{b}-2\mathbf{a}$; $\mathbf{b}-2\mathbf{a}$; $-2\mathbf{a}$; $-\mathbf{a}-\mathbf{b}$.

9. $\mathbf{p}+\mathbf{q}+\mathbf{r}+2\mathbf{s}=0$. **10.** $\overrightarrow{BC}=(3, 6, -3)-(4, 3, 2)=(-1, 3, -5)$ and hence $|\overrightarrow{BC}| = \sqrt{35}$. Similarly $|\overrightarrow{CA}| = \sqrt{68}$; $|\overrightarrow{AB}| = \sqrt{11}$. Direction cosines of $\overrightarrow{BC}=(-1/\sqrt{35}, 3/\sqrt{35}, -5/\sqrt{35})$. Similarly others.

11. Sides are $\sqrt{18}$, 3, 3. Also $BC^2=CA^2+AB^2$; hence isosceles right triangle. Direction cosines $(-1/\sqrt{18}, 1/\sqrt{18}, -4/\sqrt{18})$ for $\overrightarrow{BC}$. Similarly others.

13. Verify : $|\overrightarrow{AB}| + |\overrightarrow{BC}| = |\overrightarrow{AC}|$; also $\overrightarrow{AB} \parallel \overrightarrow{BC}$.

14. $\mathbf{p}=\dfrac{3\mathbf{c}+2\mathbf{b}}{3+2}=\frac{1}{5}(-17, 24, -5)$, $\mathbf{q}=\dfrac{-3\mathbf{c}+2\mathbf{b}}{-3+2}=(-37, 12, -13)$. Hence $\overrightarrow{AP}=\frac{1}{5}(-32, 14, -5)$; $\overrightarrow{AQ}=(-40, 10, -13)$. Now obtain the direction cosines.

15. $\frac{1}{16}(38, 57, 17)$. By the condition of the problem $BD : DC=BA : AC$. Obtain $|\overrightarrow{BA}| =3$, $|\overrightarrow{AC}| =13$. Now use the formula of section ratio.

16. (i) If $\mathbf{a}$ and $\mathbf{b}$ are collinear then it is possible to write $\mathbf{a}=k\mathbf{b}$, *i.e.* $(a_1, a_2, a_3)=(kb_1, kb_2, kb_3)$. From the definition of equality of two vectors we get $a_1=kb_1$, $a_2=kb_2$, $a_3=kb_3$; hence etc.

(ii) $\overrightarrow{AB}=(-1, -1, 1)$, $\overrightarrow{AC}=-2(-1, -1, 1)$ and so $\overrightarrow{AC}=-2\overrightarrow{AB}$ *i.e.*, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are collinear and clearly they are coinitial ; hence A. B, C are collinear.

(iii) If the required ratio be k then $\mathbf{b}=\dfrac{1}{k+1}(k\mathbf{c}+\mathbf{a})$ gives

$(1, 4, -3)=\dfrac{1}{k+1}(4k+2, 7k+5, -6k-4)$, whence $1=\dfrac{4k+2}{k+1}$ *i.e.*, $k=-\dfrac{1}{3}$.

(iv) From (iii) it follows D divides AC in the ratio 1 : 3 and now obtain the coördinates of D.

17. 1 : 2 ; $\mathbf{d}=(2, 10, 11)$.

18. If coplanar, then either AB and CD intersect or they are parallel. In fact, see that there exists a point P which is common to the lines AB and CD.

20. $AC : CB=-3 : 1$; $BD : DA=-2 : 1$: hence proceed.

22. (3/13, 12/13, 4/13).

1·10. Centroids.

DEFINITION. If **a, b, c, ...l** be the position vectors of n given points with reference to some origin O then the point G, whose position vector is given by

$$\overrightarrow{OG} = \frac{1}{n}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\cdots+\mathbf{l}),$$

is called the *centroid* (or *Mean centre* or *Centre of mean position*) of the given points.

Generalised Definition. If a point P (whose position vector is $\mathbf{p}$) be associated with a real number m, we speak of P as a *weighted point* ; m is called the *weight*.

Suppose **a, b, c, ...l** are the position vectors of n points relative to some origin O. We associate weights (*i.e.*, real numbers) $m_1, m_2, \ldots\ldots m_n$ (where $m_1+m_2+\cdots+m_n \neq 0$), one with each point in the order in which they have been written. The point G given by

$$\mathbf{g} = \overrightarrow{OG} = \frac{m_1\mathbf{a}+m_2\mathbf{b}+\cdots+m_n\mathbf{l}}{m_1+m_2+\cdots+m_n} = \frac{\Sigma m\mathbf{a}}{\Sigma m},$$

is called the *Centroid of the n weighted points* (or Centroid of the given points **a, b, c, ...l** associated with numbers $m_1, m_2, \ldots m_n$).

Corollary 1. If $m_1 = m_2 = \cdots = m_n$ then we obtain the centroid of n equally weighted points ; this point is called the *mean centre* or *centre of mean position* or simply the *centroid* of the given n points.

Corollary 2. The centroid G of two given points **a** and **b** is given by $\overrightarrow{OG} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ *i.e.*, it bisects the join of **a** and **b**.

Corollary 3. The centroid G of two weighted points **a** and **b** with weights p and q is given by $\overrightarrow{OG} = (p\mathbf{a}+q\mathbf{b})/(p+q)$ *i.e.*, it divides the join of **a** and **b** in the ratio $q : p$.

1·10·1. Theorems connected with centroids.

Theorem 1. *The centroid of n given points associated with n weights is independent of the origin.*

Proof. With reference to an origin O, suppose the position vectors of n given points are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ...$\mathbf{l}$, and relative to a different origin O' let their position vectors be $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, ...$\mathbf{l}'$. Let the n associated weights be m_1, m_2, m_3, ...m_n. Assume $\overrightarrow{OO'} = \mathbf{p}$. Then obviously (See Fig. 1.19), $\mathbf{a}' = \mathbf{a} - \mathbf{p}$.

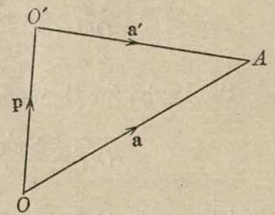


Fig. 1.19. Centroid independent of origin

Similarly $\mathbf{b}' = \mathbf{b} - \mathbf{p}$, $\mathbf{c}' = \mathbf{c} - \mathbf{p}$, ...$\mathbf{l}' = \mathbf{l} - \mathbf{p}$.

If G, G' be respectively the centroids with reference to O and O' as origin, we have

$$\overrightarrow{O'G'} = \frac{m_1\mathbf{a}' + m_2\mathbf{b}' + m_3\mathbf{c}' + \cdots + m_n\mathbf{l}'}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$= \frac{m_1(\mathbf{a} - \mathbf{p}) + m_2(\mathbf{b} - \mathbf{p}) + \cdots + m_n(\mathbf{l} - \mathbf{p})}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{m_1\mathbf{a} + m_2\mathbf{b} + \cdots + m_n\mathbf{l}}{m_1 + m_2 + \cdots + m_n} - \mathbf{p}$$

$$= \overrightarrow{OG} - \overrightarrow{OO'} = \overrightarrow{O'O} + \overrightarrow{OG} = \overrightarrow{O'G}.$$

Thus G and G' coincide which proves the theorem.

Important observation.

The theorem suggests that the position of G is unique. Thus relative to any origin the centroid of a set of weighted points has a position vector that equals the weighted average of the given set of position vectors.

Theorem 2. *If G be the centroid of a system of points* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,...$\mathbf{l}$ *with associated numbers x, y, z,...t and G' be the*

centroid of another system of points $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, $\ldots\mathbf{l}'$ *with associated numbers* x', y', z', $\ldots t'$ *then the centroid of all the points of the two system is the centroid of the points* G *and* G' *with associated numbers* $x+y+z+\cdots+t$ *and* $x'+y'+z'+\cdots+t'$.

Proof : Take O as the origin. Then for the first system

$$\overrightarrow{OG}=\frac{x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\cdots+t\mathbf{l}}{x+y+z+\cdots+t}=\frac{\Sigma x\mathbf{a}}{\Sigma x} \qquad \cdots \quad (1)$$

Similarly, for the second system of points

$$\overrightarrow{OG'}=\frac{\Sigma x'\mathbf{a}'}{\Sigma x'} \qquad \cdots \qquad \cdots \quad (2)$$

The centroid H of the two points G and G' with associated numbers Σx and $\Sigma x'$ will be given by

$$\overrightarrow{OH}=\frac{\Sigma x.\overrightarrow{OG}+\Sigma x'.\overrightarrow{OG'}}{\Sigma x+\Sigma x'}=\frac{\Sigma x\mathbf{a}+\Sigma x'\mathbf{a}'}{\Sigma x+\Sigma x'}$$

($\because$ by (1), $\overrightarrow{OG}.\Sigma x=\Sigma x\mathbf{a}$ and by (2), $\overrightarrow{OG'}.\Sigma x'=\Sigma x'\mathbf{a}'$)

The last relation shows that H is the centroid of the combined system of points.

Corollary. If a number of sub-systems are given, the centroid of the combined system of points may be calculated by considering each sub-system to be a point (*viz.*, its centroid) with associated weights Σx for that sub-system.

Theorem 3. *A linear relation of the form*

$$m_1\mathbf{p}_1+m_2\mathbf{p}_2+\cdots+m_n\mathbf{p}_n=\mathbf{0}, \qquad \ldots \quad (1)$$

connecting the position vectors of points $P_1, P_2, \ldots P_n$ *will be independent of the position of the origin* O *when and only when the sum of the scalar coefficients is zero* : *that is,*

$$m_1+m_2+\cdots+m_n=0. \qquad \ldots \quad (2)$$

Proof. As in Theorem 1 change the origin from O to O'. We write $\overrightarrow{OP_i}=\mathbf{p}_i$, $\overrightarrow{O'P_i}=\mathbf{p}'_i$, $\overrightarrow{OO'}=\mathbf{d}$. We have then $\mathbf{p}_i=\mathbf{d}+\mathbf{p}'_i$. Hence from (1)

$$(m_1+m_2+\cdots+m_n)\,\mathbf{d}+m_1\mathbf{p}'_1+m_2\mathbf{p}'_2+\cdots+m_n\mathbf{p}'_n=\mathbf{0}$$

which is of the same form as (1) if and only if (2) holds.

Corollary 1. Such linear relations for two points will imply that the points are coincident ; for three points they will mean that the points are collinear and for four points, that they are coplanar.

Corollary 2. A set of n weighted points $\mathbf{p}_i$ with weights m_i ($i=1, 2, 3, \dots n$) which satisfy a linear relation

$$\Sigma m_i\,\mathbf{p}_i=\mathbf{0},\ \ \Sigma m_i=0,$$

independent of the origin, has the intrinsic property that any point of the set is the centroid of the remaining points. For example,

$$\mathbf{p}_1=\frac{m_2\mathbf{p}_2+m_3\mathbf{p}_3+\cdots+m_n\mathbf{p}_n}{m_2+m_3+\cdots+m_n}.$$

Such sets are called *self-centroidal* sets. Theorems 1 and 2 of art. 1·72 furnish examples of self-centroidal sets.

1·10·2. Centre of mass.

In our previous discussions the weights were supposed to be real numbers, but in applications they may be any scalar quantity like length, area, volume, mass etc. Thus the notion of centroid of *weighted points* helps to correlate a number of different situations.

(1) The *centre of mass* of a number of particles of masses $m_1, m_2, m_3, \dots$ situated at points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3, \dots$ respectively is the centroid of these points associated with numbers $m_1, m_2, m_3, \dots$ If we denote the centre of mass (c.m.) by $\bar{\mathbf{a}}$, then

$$\bar{\mathbf{a}}=\frac{\Sigma m\mathbf{a}}{\Sigma m} \qquad \dots \quad (1)$$

(2) Suppose now a system of like parallel forces whose magnitudes are proportional to the masses $m_1, m_2, m_3, \ldots$ act at the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$ respectively.

The point, where the resultant force acts is called the centre of gravity of the system of particles. It can be shown that the centre of mass (c.m.) as defined before coincides with the centre of gravity (c.g.) of the system of particles.

Being a centroid, the c.m. (and c.g.) is independent of the origin. Also if the system of particles be divided into n sub-systems the c.m. of the whole system is the c.m. of n particles placed at the c.m. of each sub-system and in each case mass being equal to the total mass of the sub-system.

Take a system of axes (not necessarily rectangular) with O as origin and let the unit vectors along them be $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$. Then, the position vector of a particle of mass m is given by

$$\mathbf{a} = x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + x\hat{\mathbf{c}},$$

where (x, y, z) are the coördinates of the particle and the position vector of the c.m. will be

$$\bar{\mathbf{a}} = \bar{x}\,\hat{\mathbf{a}} + \bar{y}\,\hat{\mathbf{b}} + \bar{z}\,\hat{\mathbf{c}},$$

$(\bar{x}, \bar{y}, \bar{z})$ being the coördinates of the c.m.

Then from (1), $\bar{x}\,\hat{\mathbf{a}} + \bar{y}\,\hat{\mathbf{b}} + \bar{z}\,\hat{\mathbf{c}} = \dfrac{\Sigma m(x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + z\hat{\mathbf{c}})}{\Sigma m}.$

Equating the components of two equal vectors,

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}.$$

These equations determine the position of centre of mass.

(3) *Centroid of area, centroid of volume or more generally c.m. of a continuous distribution of matter* are defined almost from the same principle. We discuss one of them, say the *centroid of area.*

We divide the surface of any figure into n small elements. Consider one point in each element. With each point, associate

a number proportional to the area of that element. These points with assigned associated numbers will have a centroid, say G. If now $n \to \infty$ so that each element will approach to a point. The limiting position of G will now be the *centroid of the area* of the figure.

For defining the centroid of volume, we are to consider a solid figure and *replace the term area by volume.* In the general case, we shall consider either a surface distribution or a volume distribution.

Examples. I(C)

1. The centroid of the vertices of a triangle trisects the medians.

2. Let the points P, Q, R divide respectively the sides BC, CA, AB of a triangle ABC in the ratio $k:1$ in every case. Prove that the centroid of P, Q, R will coincide with that of A, B, C.

3. Masses proportional to 1, 4, 6, 8 are placed respectively at the extremities of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ which conform to the relations $5\mathbf{a}+2\mathbf{b}+18\mathbf{c}=\mathbf{0}$ and $\mathbf{a}-5\mathbf{b}-12\mathbf{d}=\mathbf{0}$. Prove that the mass-centre coincides with the origin of vectors.

4. Find the centroid of the $3n$ points :

$$\mathbf{i},\ 2\mathbf{i},\ 3\mathbf{i},\ldots\ldots n\mathbf{i}\ ;\ \mathbf{j},\ 2\mathbf{j},\ 3\mathbf{j},\ldots\ldots n\mathbf{j}\ ;\ \mathbf{k},\ 2\mathbf{k},\ 3\mathbf{k},\ldots\ldots n\mathbf{k}.$$

5. Particles of equal masses are placed at $(n-2)$ of the corners of a regular polygon of n sides. Find their centre of mass.

6. A line AB is bisected at P_1, P_1B at P_2, P_2B at P_3 and so on, *ad infinitum* ; and particles of masses m, $\frac{1}{2}m$, $\frac{1}{4}m$, $\frac{1}{8}m$,... are placed at the points P_1, P_2, P_3,... . Prove that the distance of their centre of mass from B is equal to one-third of the distance from A to B.

7. Find the centre of mass of particles of masses 1, 2, 3, 4, 5 6, 7, 8 gms. respectively, placed at the corners of a unit cube ; the first four at the corners A, B, C, D of one face and the last four at their projections A', B', C', D' respectively on the opposite face.

8. Three points A, B, C are given by $(5, -3, -4)$, $(2, -3, -1)$ and $(2, -3, 3)$. Find their centroid. Also find the centroid if masses 4, 5, 3 respectively be associated with them.

9. $\overrightarrow{A_rB_r}$ $(r=1, 2, 3, \ldots n)$ are n vectors and m_r be n real numbers such that $\Sigma m_r = M$. Prove that

$$\Sigma m_r \overrightarrow{A_rB_r} = M.\overrightarrow{GG'},$$

where G is the centroid of the points A_r with associated numbers m_r and G' is the centroid of B_r with associated numbers m_r.

10. Let A, B, C, D be the vertices of tetrahedron and P be their mean centre. If A', B', C', D' are the mean centres of the triads BCD, CDA, DAB, ABC show that P divides each of the segments AA', BB', CC', DD' in the ratio 3 : 1.

Hints and Answers

1. If G be the centroid relative to the origin O then $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are position vectors of the vertices. Writing $\overrightarrow{OG} = \frac{1}{3}\left[\mathbf{a} + 2 \cdot \frac{\mathbf{b}+\mathbf{c}}{2}\right]$ it is easy to see that G is the point of trisection of the medians.

2. $\mathbf{p} = \frac{k\mathbf{c}+\mathbf{b}}{k+1}$, $\mathbf{q} = \frac{k\mathbf{a}+\mathbf{c}}{k+1}$, $\mathbf{r} = \frac{k\mathbf{b}+\mathbf{a}}{k+1}$. Now see that $\frac{1}{3}(\mathbf{p}+\mathbf{q}+\mathbf{r}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$; hence etc.

3. $\overrightarrow{OG} = \frac{1}{19}(\mathbf{a}+4\mathbf{b}+6\mathbf{c}+8\mathbf{d}) = \mathbf{0}$ (using the given relations) ; hence etc.

4. $\frac{n+1}{6}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

5. If there were masses at all corners, the c.m. would be the centroid O of the polygon. Suppose the vacant corners are A and B where equal masses have been placed in order to get the c.m. at O. Now the c.m. of equal masses at A and B will be at N, the mid-point of AB.

Suppose the required c.m. be at G. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of O, N, G. Then

$$\mathbf{a}=\frac{2\mathbf{b}+(n-2)\mathbf{c}}{2+n-2}; \quad \text{whence } \mathbf{c}=\frac{n\mathbf{a}-2\mathbf{b}}{n-2}.$$

Thus the required point G lies on the join of O and N so that $NO : OG = n-2 : 2$.

7. Take A as origin. The position vectors of A, B, C, D ; A', B', C', D' are respectively $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$; $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$.

$$\therefore \quad \overrightarrow{GA}=\frac{1.(0, 0, 0)+2(1, 0, 0)+\cdots+8(0, 1, 1)}{1+2+\cdots+8}.$$

$\therefore$ position vector of $G=(\frac{1}{2}, \frac{11}{18}, \frac{13}{18})$.

8. $(3, -3, -\frac{2}{3})$; $(3, -3, -1)$.

Summary of Chapter 1.

1. A *Vector* is a directed line segment : A directed line segment has three characteristics—length, support and sense. Length of $\mathbf{a}$ is denoted by $|\mathbf{a}|$ or by a.

2. Two vectors are *equal* if they have same length and sense but same or parallel supports. Vectors that conform to this definition of equality are *free* vectors. If their supports are always confined to a definite line then they are called *line vectors*. Two line vectors are *equal* when and only when they have the same length and same sense and *lie on the same support.*

3. A *null vector* (or zero vector) $\mathbf{0}$ has length zero but no definite direction while a *proper vector* has a definite length and direction. A vector of length unity is called a *unit vector* ; unit vector of $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$; $|\hat{\mathbf{a}}|=1$ and $\hat{\mathbf{a}}$ has the same direction as $\mathbf{a}$.

4. Two vectors are *collinear* if they have the same or parallel supports.

5. Vectors are added by triangle construction (Fig. 1.3). Vector addition is

commutative : $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$,

associative : $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$.

6. The negative of $\mathbf{a}$ (denoted by $-\mathbf{a}$) is a vector of same length as $\mathbf{a}$ but of opposite direction.

7. The difference of two vectors is defined by

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$$

or by $(\mathbf{a}-\mathbf{b})+\mathbf{b}=\mathbf{a}.$

8. Vector equations :

$$\mathbf{a}+\mathbf{x}=\mathbf{b} \quad \text{gives} \quad \mathbf{x}=\mathbf{b}-\mathbf{a}$$
$$\mathbf{a}+\mathbf{b}=\mathbf{0} \quad \text{gives} \quad \mathbf{b}=-\mathbf{a}.$$

9. Scalar multiple of a vector :

$$\text{If } m>0, \quad \pm m\mathbf{a} \text{ has } \begin{cases} \text{length} & m|\mathbf{a}| \\ \text{direction} & \pm\mathbf{a} \end{cases}$$

whence, $\quad m(n\mathbf{a})=(mn)\mathbf{a}$; $(m+n)\mathbf{a}=m\mathbf{a}+n\mathbf{a}$;
$m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}.$

10. Vectors may be added, subtracted and multiplied by scalars in conformity with the laws of algebra of real numbers.

11. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear if and only if a suitable scalar x can be found such that $\mathbf{a}=x\mathbf{b}$.

12. With respect to an origin O, any point P of space is determined by its position vector $\overrightarrow{OP}$ ($\mathbf{p}$). Any vector $\overrightarrow{AB}$ may be expressed in terms of the position vectors of its ends :

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}.$$

13. If A, B, C are points of a straight line, C is said to divide the segment AB in the ratio $k:1$ if $AC=kCB$; whence the position vector of C relative to an origin O is given by

$$\mathbf{c}=\frac{\mathbf{a}+k\mathbf{b}}{1+k}; \quad \mathbf{c}=\frac{m\mathbf{a}+n\mathbf{b}}{m+n} \quad \text{if } k=\frac{n}{m}.$$

14. Three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ will lie on a line *when and only when*, there exist three numbers x, y, z not all zero, such that

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{0}, \quad x+y+z=0.$$

15. Four points **a**, **b**, **c**, **d**, no three collinear, will lie in a plane *when, and only when*, there exist four scalars x, y, z, t, not all zero, such that

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+t\mathbf{d}=\mathbf{0}\ ;\ x+y+z+t=0.$$

16. If we adopt a right-handed set of rectangular axes $O-xyz$ as a system of reference (Fig. 1.18), any free vector $\mathbf{r}$ may be shifted to become the position vector $\overrightarrow{OP}$ with its initial point at the origin. Then, if P has the rectangular coördinates (x, y, z) we say $\mathbf{r}=\overrightarrow{OP}$ has the *components* x, y, z along the directions of the axes. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors in the positive directions of x, y, z we have

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

17. $\mathbf{r}$ is completely known by its components (x, y, z), a *number-triple* written in a definite order. Thus we may regard number-triple as a vector and write

$$\mathbf{r}=(x,\ y,\ z)$$

where $|\mathbf{r}|^2=x^2+y^2+z^2$; the direction of $\mathbf{r}$ is given by its direction cosines : $x=|\mathbf{r}|\cos\alpha, y=|\mathbf{r}|\cos\beta, z=|\mathbf{r}|\cos\gamma$.

18. In the language of number-triples :

$$\mathbf{0}=(0, 0, 0)\ ;\ \mathbf{i}=(1, 0, 0)\ ;\ \mathbf{j}=(0, 1, 0)\ ;\ \mathbf{k}=(0, 0, 1).$$

If $\mathbf{a}=(a_1, a_2, a_3)$, $\mathbf{b}=(b_1, b_2\ b_3)$ then

$$\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2,+b_2, a_3+b_3)\ ;$$

$$\mathbf{a}-\mathbf{b}=(a_1-b_1, a_2-b_2, a_3-b_3)\ ;$$

$$k\mathbf{a}=(ka_1, ka_2, ka_3)\ ;\ \hat{\mathbf{a}}=(\cos\alpha, \cos\beta, \cos\gamma).$$

19. The *centroid* G of n weighted points P_i with weights m_i $(i=1, 2,\dots n)$ is defined by

$$\overrightarrow{OG}=\frac{\Sigma m_i\mathbf{p}_i}{\Sigma m_i}\ ;$$

$\mathbf{p}_i$ is the position vector of P_i relative to O.

The *centroid* G of n equally weighted points P_i is called their *mean centre* or simply *centroid of the given points* given by

$$\overrightarrow{OG} = \frac{\Sigma \mathbf{p}_i}{n}.$$

20. (i) The centroid of a set of points (or a set of weighted points) is *unique* in the sense that it is independent of the origin : relative to any origin it has a position vector that equals the weighted average of the given set of position vectors.

(ii) In finding the centroid of two or more sets of points, each set may be replaced by its weighted centroid.

21. A linear relation

$$m_1\mathbf{p}_1 + m_2\mathbf{p}_2 + \cdots + m_n\mathbf{p}_n = \mathbf{0},$$

connecting the position vectors of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_n$ will be independent of origin if and only if the scalar coefficients is zero *i.e.*,

$$m_1 + m_2 + \cdots + m_n = 0.$$

For two, three and four points such linear relations have the respective meanings : the points are coincident, collinear and coplanar.

For more than four points the relation will mean that any point $\mathbf{p}_i$ of the set is the centroid of the remaining weighted points (provided $m_i \neq 0$).

2

Applications of Elementary Operations : Geometrical and Physical

2·1. Parametric equation to a straight line.

A. *To find the vectorial equation of a line which passes through a given point and is parallel to a given vector.*

Suppose the given line is parallel to the vector **b** and passes through a point A (whose position vector relative to an origin O is **a**). Let P be any point on the line (Fig. 2.1). We denote its position vector with reference to O by **r**. Since P may be any point on the line, we call **r** the *current vector*.

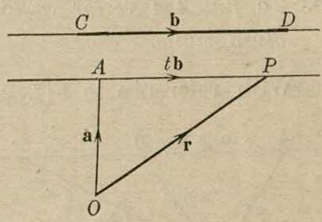


Fig. 2.1. Parametric equation of a line

The vector $\overrightarrow{AP}$ is evidently parallel to **b** and hence can be expressed as $t\mathbf{b}$ where t is a real number (positive or negative according as P is on the right or left of A). Hence

$$\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\mathbf{a}+t\mathbf{b}. \qquad \dots \quad (1)$$

Now observe that for any real value of t there corresponds a point P on the line and conversely, any point on the given line has a position vector given by (1) for some value of t. Thus we speak of (1) as the vector equation of the line ; t is called a scalar parameter and this form of the equation is called parametric form.

Corollary 1. The equation of a line passing through the origin and parallel to **b** will be $\mathbf{r}=t\mathbf{b}$. For, A coincides with O so that $\mathbf{a}=\mathbf{0}$.

Corollary 2. If with reference to a set of rectangular system of axes through O, the coördinates of P and A be respectively (x, y, z) and (a_1, a_2, a_3) and if the vector $\mathbf{b}=(b_1, b_2, b_3)$ then (1) will give

$$(x, y, z)-(a_1, a_2, a_3)=t(b_1, b_2, b_3)=(tb_1, tb_2, tb_3)$$

and hence,

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=t \qquad \cdots \quad (2)$$

which is the familiar equation (of three-dimensional geometry) to a straight line passing through the point (a_1, a_2, a_3) with direction cosines proportional to (b_1, b_2, b_3).

B. *To find the vector equation of a line passing through two given points.*

With reference to Fig. 2.2, let A and B be two given points whose position vectors relative to O are $\mathbf{a}$ and $\mathbf{b}$. As before, let P be any point on the line, its position vector being $\mathbf{r}$. It is clear that $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$. We may now consider this problem as to find the equation to a line passing through the point $\mathbf{a}$ and parallel to vector $\overrightarrow{AB}$. Hence, its equation will be

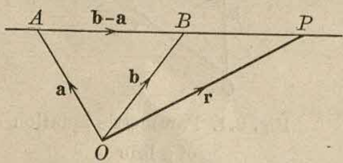


Fig. 2.2. Equation of a line through $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})$$

$$\text{or,} \quad \mathbf{r}=(1-t)\mathbf{a}+t\mathbf{b}. \qquad \cdots \quad (3)$$

Writing $1-t=s$, the required equation may also be written in the form :

$$\mathbf{r}=s\mathbf{a}+(1-s)\mathbf{b}. \qquad \cdots \quad (4)$$

Corollary 3. If A and B have the coördinates (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively with reference to a set of rect-

angular axes (origin O) and (x, y, z) be the coördinates of P, then (3) reduces to

$$(x, y, z) = (1-t)(a_1, a_2, a_3) + t(b_1, b_2, b_3),$$

whence, $$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3} = t,$$

the familiar cartesian equation to a line passing through two points (a_1, a_2, a_3) and (b_1, b_2, b_3).

An Important Note.

If we re-write the equation (3) as

$$(1-t)\mathbf{a} + t\mathbf{b} - \mathbf{r} = \mathbf{0},$$

we find the sum of the scalar coefficients *viz.*, $(1-t)+t+(-1)=0$.

Thus, for three points A, B, P to be collinear, it is *necessary* that there should exist a linear relation, where the sum of the scalar coefficients is zero.

This conditon is also sufficient.

For if we assume a linear relation

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{r} = \mathbf{0}, \text{ where } x+y+z=0,$$

then we may construct

$$\mathbf{r} = -\frac{y}{z}\mathbf{b} - \frac{x}{z}\mathbf{a} = \frac{y}{x+y}\mathbf{b} + \frac{x}{x+y}\mathbf{a}$$

$$= t\mathbf{b} + (1-t)\mathbf{a};$$

where $t = y/(x+y)$. This implies that the three points are collinear.

This is the alternative proof of Theorem 1 of art. 1·72.

2·2. Parametric equation of a plane.

A. *To find the vectorial equation of a plane which passes through a given point and is parallel to two given lines.*

With reference to the origin O, let the position vector of

the given point A be $\mathbf{a}$ and let the position vector of *any point* P in the plane be $\mathbf{r}$ (Fig. 2.3). Let the two given lines to which the plane is parallel are given by the vectors $\mathbf{b}$ and $\mathbf{c}$. Obviously the three vectors $\overrightarrow{AP}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar. Hence

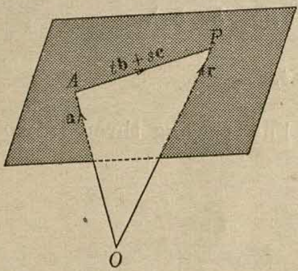


Fig. 2.3. Equation of a plane through $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$

$$\overrightarrow{AP} = t\mathbf{b} + s\mathbf{c},$$

where t and s are suitable scalars. Thus,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

i.e., $$\mathbf{r} = \mathbf{a} + t\mathbf{b} + s\mathbf{c} \quad \cdots \quad (1)$$

For every point P of the plane, there exist some values of t and s ; and conversely. Further no point outside the plane can be represented by (1). Hence (1) is the required equation of the plane ; t and s are called parameters and $\mathbf{r}$, the current vector.

Corollary 1. To find the equation to the plane passing through the origin and parallel to $\mathbf{b}$ and $\mathbf{c}$, we put $\mathbf{a} = \mathbf{0}$ and obtain $\mathbf{r} = t\mathbf{b} + s\mathbf{c}$ as the required equation.

Corollary 2. If $\overrightarrow{OP} = (x, y, z)$, $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$; then we may write (1) in the form :

$$(x, y, z) = (a_1 + tb_1 + sc_1, a_2 + tb_2 + sc_2, a_3 + tb_3 + sc_3).$$

Equating the corresponding components of the two equal vectors, we may deduce the equation in the form $Ax + By + Cz + D = 0$ which occurs frequently in three-dimensional coördinate geometry ; A, B, C, D are suitably chosen constants.

B. *To find the equation of a plane through two given points and parallel to a given line.*

Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of two given points A and B with reference to an origin O (Fig. 2.4). Let $\mathbf{c}$ be the vector

parallel to the plane. Then the vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ is also parallel to the plane. Thus, the equation of the required plane is a plane

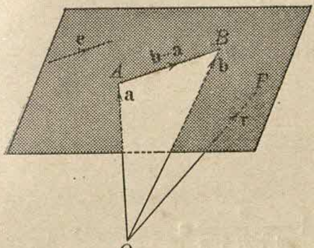


Fig. 2.4. Equation of a plane through **a**, **b** and parallel to **c**

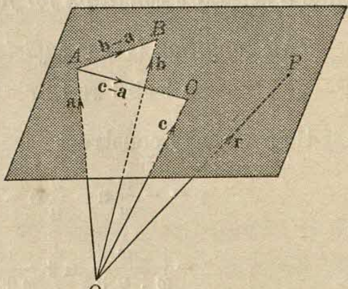


Fig. 2.5. Equation of a plane through **a**, **b**, **c**

through **a** and parallel to **c** and $\mathbf{b} - \mathbf{a}$. Hence its equation will be

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + t(\mathbf{b} - \mathbf{a}) + s\mathbf{c} \\ &= (1-t)\mathbf{a} + t\mathbf{b} + s\mathbf{c} \qquad \cdots \qquad \cdots \quad (2) \end{aligned}$$

C. *To find the equation of a plane through three given points whose position vectors are* **a**, **b** *and* **c**.

From Fig. 2.5 it is clear that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane. Hence the required equation is same as the equation of a plane passing through A and parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and so its equation is

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + t(\mathbf{b} - \mathbf{a}) + s(\mathbf{c} - \mathbf{a}) \\ &= (1-t-s)\mathbf{a} + t\mathbf{b} + s\mathbf{c}. \qquad \cdots \quad (3) \end{aligned}$$

An Important Note.

The four coplanar points A, B, C, P are related by

$$\mathbf{r} = (1-t-s)\mathbf{a} + t\mathbf{b} + s\mathbf{c},$$

or, $$(1-t-s)\mathbf{a} + t\mathbf{b} + s\mathbf{c} - \mathbf{r} = \mathbf{0},$$

where the sum of the scalar coefficients $(1-t-s)+t+s+(-1)=0$. Thus there exists a linear relation connecting their position vectors, the sum of whose scalar coefficients is zero. This is thus *a necessary condition for four points to be coplanar.*

This condition is also sufficient. For, if

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+u\mathbf{r}=\mathbf{0}, \text{ where } x+y+z+u=0$$

then we may construct

$$\begin{aligned}\mathbf{r} &= -\frac{x}{u}\mathbf{a}-\frac{y}{u}\mathbf{b}-\frac{z}{u}\mathbf{c}\\ &= \frac{x}{x+y+z}\mathbf{a}+\frac{y}{x+y+z}\mathbf{b}+\frac{z}{x+y+z}\mathbf{c}\\ &= (1-t-s)\mathbf{a}+t\mathbf{b}+s\mathbf{c},\end{aligned}$$

where $$t=\frac{y}{x+y+z} \text{ and } s=\frac{z}{x+y+z}.$$

This implies that the four points **a**, **b**, **c**, **r** are coplanar.

This is the alternative proof of Theorem 2, art. 1·72.

Examples. II(A)

1. Write down the equation of a line through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

2. Find the equation of the line joining the points $(2, -3, -1)$ and $(8, -1, 2)$.

3. Find the equation of the line through the point $(-2, 3, 4)$ and parallel to the vector $(2, 3, 4)$.

4. Show that the line through $P(4, -3, -1)$ and parallel to $(1, 4, 7)$ is

$$\frac{x-4}{1}=\frac{y+3}{4}=\frac{z+1}{7}$$

and find the two points on it at a distance of $\sqrt{1056}$ from P.

5. In the line

$$\frac{x-3}{9}=\frac{y+4}{6}=\frac{z+2}{2},$$

find two points each of whose distance from $(3, -4, -2)$ is 22.

6. Write down the equation of a plane through $P(2, 2, -1)$, $Q(3, 4, 2)$ and $R(7, 0, 6)$.

7. Write down the equation of a plane through three points $(0, 0, 0)$, $(2, 4, 1)$ and $(4, 0, 2)$.

8. Find the equation of the line joining the points $(7, -3, 4)$ and $(2, -1, 1)$ and determine the point where it cuts the plane through $(2, 1, -3)$, $(4, -1, 2)$ and $(3, 0, 1)$.

9. Prove that the line

$$\frac{x+1}{3}=\frac{y+3}{3}=\frac{z+5}{7}$$

intersects the plane through $(0, 0, 0)$, $(1, 2, -5)$ and $(2, 3, -1)$ at the point $-\frac{2}{5}(31, 36, 79)$.

Hints and Answers

1. With reference to an assigned origin O, let $\overrightarrow{OP}=(x_1, y_1, z_1)$, $\overrightarrow{OQ}=(x_2, y_2, z_2)$. The required equation will then be

$$\mathbf{r}=\overrightarrow{OP}+t\overrightarrow{PQ}$$

i.e., $(x, y, z)=(x_1, y_1, z_1)+t(x_2-x_1, y_2-y_1, z_2-z_1)$.

Hence the equation in the cartesian form is,

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

2. $(x-2)/6=(y+3)/2=(z+1)/3$.

3. $(x+2)/2=(y-3)/3=(z-4)/4$.

4. *Second part.* Take a point $Q=(x, y, z)$ on the line at a distance of $\sqrt{(1056)}$ from P.

Then we have

$$\overrightarrow{PQ}=(x-4,\ y+3,\ z+1)\ ;$$

$$|\ \overrightarrow{PQ}\ |^2=(x-4)^2+(y+3)^2+(z+1)^2$$
$$=66k^2, \text{ each ratio of the equation of the line being } k.$$

$$\therefore\quad 66k^2=1056\ ;\ \ i.e.,\ k=\pm 4.$$

Thus the two points are $(8, 13, 27)$ and $(0, -19, -29)$.

5. $(-15, -16, -6)$ and $(21, 8, 2)$.

6. Using notations similar to Ex. 1. above, and writing

$$\mathbf{r}=(1-s-t)\overrightarrow{OP}+s\,\overrightarrow{OQ}+t\,\overrightarrow{OR}$$

obtain $(x, y, z)=(1-s-t)(2, 2, -1)+s(3, 4, 2)+t(7, 0, 6)$. Equate the corresponding components. Eliminating s and t obtain $5x+2y-3z=17$, the required plane.

7. $x-2z=0$ is the equation of the required plane.

8. The equation of the line is

$$(x, y, z)=(7, -3, 4)+t(2, -1, 1)$$

The equation of the plane is

$$(x, y, z)=(1-u-v)(2, 1, -3)+u(4, -1, 2)+v(3, 0, 1).$$

For the common point, (x, y, z) have the same values. Hence equating corresponding components we get

$$7+2t=2(1-u-v)+4u+3v$$
$$-3-t=(1-u-v)-u$$
$$4+t=-3(1-u-v)+2u+v,$$

whence, from first two, on addition we obtain $t=-1$, so that the required point of intersection is $(5, -2, 3)$.

2·3. Equations of Bisectors of the angle between two lines.

We proceed to find the equation of the *internal bisector* of the angle between two lines OB and OA, the unit vectors along them being $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$ respectively. Take their point of intersection O as origin and let P be any point on the bisector. Draw PM parallel to AO cutting OB at M.

From Fig. 2.6 it is clear that $\angle AOP = \angle POM = \angle OPM$ and hence $OM = PM$. But $\overrightarrow{OM} = t\hat{\mathbf{b}}$ and $\overrightarrow{MP} = t\hat{\mathbf{a}}$.

($\because$ $\overrightarrow{OM} \parallel \hat{\mathbf{b}}$ and $\overrightarrow{MP} \parallel \hat{\mathbf{a}}$ and their magnitudes are same)

Then $\overrightarrow{OP} = \mathbf{r} = \overrightarrow{OM} + \overrightarrow{MP} = t(\hat{\mathbf{b}} + \hat{\mathbf{a}})$ ... (1)

For different values of t we get different points on the line OP. Thus (1) is the required equation of the bisector.

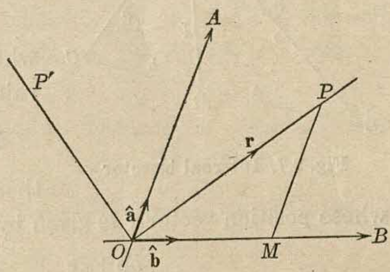


Fig. 2.6. Equations of bisectors

The *external bisector* OP' of the angle between OB and OA is the same as the internal bisector of the angle between OB' (not shown in the figure) and OA, the unit vectors along them being $-\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$ and hence the equation of OP' will be

$$\mathbf{r} = t\,(\hat{\mathbf{a}} - \hat{\mathbf{b}}) \qquad \cdots \qquad (2)$$

Corollary. If, instead of unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ along OA and OB we are given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ where the lengths of OA and OB are a and b then the equations (1) and (2) will reduce to

$$\mathbf{r} = t\left(\frac{\mathbf{a}}{a} \pm \frac{\mathbf{b}}{b}\right) \qquad (\because \quad \hat{\mathbf{a}} = \mathbf{a}/a,\ \mathbf{b} = \hat{\mathbf{b}}/b).$$

2·31. Theorems on Bisectors.

Theorem. *The internal bisector of any angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.*

Proof. Suppose AD bisects the $\angle BAC$ internally and meets the opposite side BC at D (Fig. 2.7) Let the length of sides BC, CA, AB be a, b, c respectively. Take A as origin.

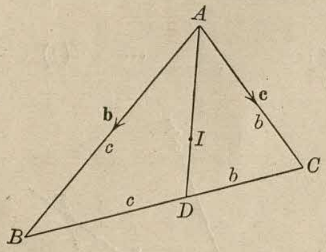


Fig. 2.7. Internal bisector

Let $\overrightarrow{AB}=\mathbf{b}$ and $\overrightarrow{AC}=\mathbf{c}$. Then the equation of AD will be

$$\mathbf{r}=t\left(\frac{1}{c}\,\mathbf{b}+\frac{1}{b}\,\mathbf{c}\right)=t\frac{b\mathbf{b}+c\mathbf{c}}{bc}$$

where t can assume arbitrary values.

Let us choose $t=\dfrac{bc}{b+c}$. Then there exists a point on the line AD whose position vector $\mathbf{r}$ is given by

$$\mathbf{r}=\frac{b\mathbf{b}+c\mathbf{c}}{b+c}.$$

But from *our knowledge of section ratio*, we may say that this point also lies on the join of $\mathbf{b}$ and $\mathbf{c}$. So this point must be the point D. It thus divides the join of $\mathbf{b}$ and $\mathbf{c}$ in the ratio of

$$c:b=AB:AC$$

This proves the theorem.

Otherwise : The point $\dfrac{b\mathbf{b}+c\mathbf{c}}{b+c}$ which lies on the bisector AD is the centroid of two points $\mathbf{b}$ and $\mathbf{c}$ with associated numbers b and c. Hence this point must lie on the join of $\mathbf{b}$ and $\mathbf{c}$, dividing it in the ratio $c:b$. (See art. 1·10, cor. 3)

Alternative method. The equation of the bisector AD is

$$\mathbf{r}=t\left(\frac{b\mathbf{b}+c\mathbf{c}}{b+c}\right)$$

and the equation of the line BC joining $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r}=\mathbf{b}+s(\mathbf{c}-\mathbf{b}).$$

At the point where AD and BC meet, the value of $\mathbf{r}$ will be the same. Thus for a common point,

$$t\left(\frac{b\mathbf{b}+c\mathbf{c}}{b+c}\right)=\mathbf{b}+s(\mathbf{c}-\mathbf{b})\ ; \quad \text{whence } t=\frac{bc}{b+c}.$$

Hence the point D has the position vector $\dfrac{b\mathbf{b}+c\mathbf{c}}{b+c}$. Now arguing as before, we may prove the theorem.

Theorem 2. *The external bisector of any angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.*

The proof is exactly similar to Theorem 1. We leave it as an exercise to the students.

Theorem 3. *The internal bisectors of the angles of a triangle are concurrent.*

Proof. As in Theorem 1, (Fig. 2.7) we can show that the position vector of D is,

$$\frac{b\mathbf{b}+c\mathbf{c}}{b+c}.$$

Let us take a point I on AD such that

$$AI : ID = b+c : a.$$

The position vector of I will then be

$$\frac{a\mathbf{a}+b\mathbf{b}+c\mathbf{c}}{a+b+c}.$$

By symmetry we can easily imagine that this point must also lie on the bisectors of the angles at B and C. Hence the bisectors are concurrent.

Note. The point I is the centroid of three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with associated numbers a, b, c respectively.

Theorem 4. *The internal bisector of the angle A and the external bisectors of angles B and C of the triangle ABC are concurrent.*

The proof is similar to Theorem 3. The point of concurrence will be the centroid of points **a**, **b**, **c** with associated numbers a, $-b$, $-c$ respectively.

Examples. II(B)

1. If the interior and exterior bisectors of the angle A of a triangle ABC meet the base BC at D and E, prove that BD, BC and BE are in harmonic progression.

2. Prove that the intersections of the bisectors of the exterior angles of any triangle with opposite sides are collinear.

3. ABC is a triangle ; AD, AD' are bisectors of the angle A meeting BC in D, D' respectively. A' is the mid-point of DD'. B', C' are similar points on CA and AB. Show that A', B', C' are collinear.

Hints and Answers

1. $\frac{BD}{BC}=\frac{c}{b+c}$; $\frac{BE}{BC}=\frac{c}{c-b}$. Hence show that $\frac{2}{BC}=\frac{1}{BD}+\frac{1}{BE}$.

2. $\mathbf{d}=\frac{b\mathbf{b}-c\mathbf{c}}{b-c}$, $\mathbf{d}'=\frac{c\mathbf{c}-a\mathbf{a}}{c-a}$, $\mathbf{d}''=\frac{a\mathbf{a}-b\mathbf{b}}{a-b}$ where $\mathbf{d}$, $\mathbf{d}'$, $\mathbf{d}''$ are points where the exterior bisectors meet the opposite sides. Next deduce

$$(b-c)\mathbf{d}+(c-a)\mathbf{d}'+(a-b)\mathbf{d}''=0,$$

a linear relation where sum of the scalar coefficients is zero ; hence collinear.

2·4. Some Important Theorems.

(A) Theorem of Menelaus. *If a transversal cuts the sides BC, CA, AB of a triangle at the points P, Q, R respectively the product of the ratios in which P, Q, R divide those sides is equal to -1 ; and conversely.*

Proof. Suppose R divides AB in the ratio $x : y$. Now choose a value z such that Q divides CA in the ratio $y : z$ (Fig. 2.8). Hence with our usual symbols for position vectors,

$$\mathbf{q}=\frac{z\mathbf{c}+y\mathbf{a}}{z+y}\text{;}\quad \mathbf{r}=\frac{y\mathbf{a}+x\mathbf{b}}{y+x}.$$

Eliminating **a**, we have

$$(z+y)\mathbf{q}-(x+y)\mathbf{r}=z\mathbf{c}-x\mathbf{b}$$

i.e., $$\frac{(z+y)\mathbf{q}-(x+y)\mathbf{r}}{(z+y)-(x+y)}=\frac{z\mathbf{c}-x\mathbf{b}}{z-x}=\mathbf{p}\ (\text{say}) \quad \cdots \quad (1)$$

which implies that **p** is the position vector of the point which divides both the lines QR and BC, *i.e.*, it is the position vector of the point P itself. Again, since

$$\mathbf{p}=\frac{z\mathbf{c}-x\mathbf{b}}{z-x},$$

P divides BC in the ratio $z:(-x)$; consequently the product of the three ratios is equal to -1.

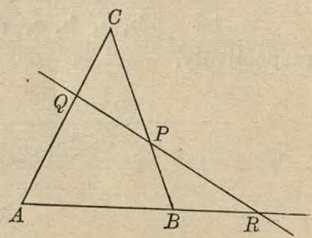


Fig. 2.8. Theorem of Menelaus

The Converse is also true.

Suppose R divides AB in the ratio $x:y$, Q divides CA in the ratio $y:z$. The product of the three ratios being equal to -1, P must divide BC in the ratio $z:(-x)$. Then,

$$\mathbf{r}=\frac{y\mathbf{a}+x\mathbf{b}}{y+x},\ \mathbf{q}=\frac{z\mathbf{c}+y\mathbf{a}}{z+y}\ \text{and}\ \mathbf{p}=\frac{z\mathbf{c}-x\mathbf{b}}{z-x},$$

$$\text{so that } (z-x)\mathbf{p}-(z+y)\mathbf{q}+(x+y)\mathbf{r}=\mathbf{0},$$

where the sum of the scalar coefficients

$$(z-x)+\{-(z+y)\}+(x+y)=0.$$

This proves that P, Q and R are collinear (see art. 1·72. Theorem 1).

(B) Theorem of Desargues. *If three concurrent straight lines OA, OB, OC are produced to D, E, F respectively then the points of intersections of AB and DE, CB and FE, AC and DF are collinear. Conversely, if the points of intersections are real and collinear, then DA, EB, FC are concurrent at O.*

Another form of the statement. *If the lines joining corresponding vertices of two triangles are concurrent, the points of intersections of the corresponding sides are collinear and conversely.*

Proof. Let the position vectors (relative to a certain origin) of

$$O, A, B, C, D, E, F, P, Q, R$$

be $\quad \mathbf{h}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{p}, \mathbf{q}, \mathbf{r}$

respectively.

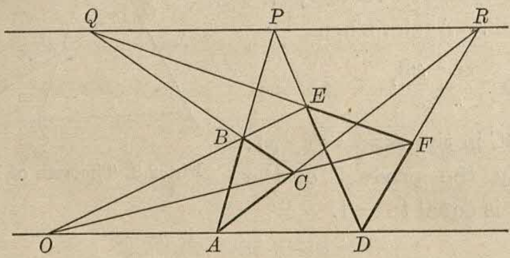


Fig. 2.9. Theorem of Desargues

Now, DA, EB and FC are concurrent at O. Therefore the position vector of O will be given by

$$\mathbf{h} = \frac{x\mathbf{a} + x'\mathbf{d}}{x + x'} = \frac{y\mathbf{b} + y'\mathbf{e}}{y + y'} = \frac{z\mathbf{c} + z'\mathbf{f}}{z + z'},$$

the scalars x, x' ; y, y' and z, z' have been suitably chosen.

We may write $\quad x\mathbf{a} + x'\mathbf{d} = y\mathbf{b} + y'\mathbf{e} = z\mathbf{c} + z'\mathbf{f} \quad \cdots \quad (1)$

where $\quad x + x' = y + y' = z + z' = 1 \quad \cdots \quad (2)$

Now, from the last two relations of (1) and (2), we find

$$y\mathbf{b} - z\mathbf{c} = z'\mathbf{f} - y'\mathbf{e}$$

and $\quad y - z = z' - y'$

$$i.e., \quad \frac{y\mathbf{b} - z\mathbf{c}}{y - z} = \frac{z'\mathbf{f} - y'\mathbf{e}}{z' - y'} = \mathbf{q}$$

(since $\mathbf{q}$ is the position vector of Q, the point of intersection of CB and FE).

Similarly, $$\frac{x\mathbf{a}-z\mathbf{c}}{x-z}=\frac{z'\mathbf{f}-x'\mathbf{d}}{z'-x'}=\mathbf{r}$$

and, $$\frac{x\mathbf{a}-y\mathbf{b}}{x-y}=\frac{y'\mathbf{e}-x'\mathbf{d}}{y'-x'}=\mathbf{p}.$$

We may now verify that

$$(x-y)\mathbf{p}-(x-z)\mathbf{r}+(y-z)\mathbf{q}=\mathbf{0},$$

where the sum of the scalar coefficients vanish. This implies that the three points **p**, **q**, **r** are collinear.

The Converse is also true.

Given P, Q, R are collinear. Let Q divide BC in the ratio $x:y$, then

$$\mathbf{q}=\frac{y\mathbf{b}+x\mathbf{c}}{y+x}.$$

Suppose R divides CA in the ratio $z:x$, then

$$\mathbf{r}=\frac{z\mathbf{a}+x\mathbf{c}}{z+x}.$$

To find P, we consider the equations of the lines AB and QR

$$\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})$$

and $$\mathbf{r}=\mathbf{q}+s\,(\mathbf{r}-\mathbf{q}).$$

For the common point, we equate the two values of **r** and then the position vector of P is easily seen to be,

$$\mathbf{p}=\frac{z\mathbf{a}-y\mathbf{b}}{z-y}.$$

Thus, we have the relation

$$(y+x)\mathbf{q}-(z+x)\mathbf{r}+(z-y)\mathbf{p}=\mathbf{0} \qquad \cdots \quad (3)$$

between the three collinear points **p**, **q** and **r**.

Again denoting by $x':y'$, $z':x'$ the ratios in which Q and R divide EF and FD, we obtain,

$$\mathbf{q}=\frac{y'\mathbf{e}+x'\mathbf{f}}{y'+x'},\quad \mathbf{r}=\frac{z'\mathbf{d}+x'\mathbf{f}}{z'+x'}.$$

Now, using similar arguments as before, we obtain,

$$\mathbf{p}=\frac{z'\mathbf{d}-y'\mathbf{e}}{z'-y'}$$

whence the relation

$$(y'+x')\mathbf{q}-(z'+x')\mathbf{r}+(z'-y')\mathbf{p}=\mathbf{0} \quad \cdots \quad (4).$$

Relations (3) and (4) give,

$$\frac{y+x}{y'+x'}=\frac{z+x}{z'+x'}=\frac{z-y}{z'-y'}=k \text{ (say)} \quad \cdots \quad (5)$$

Also
$$\left.\begin{aligned} \frac{y\mathbf{b}+x\mathbf{c}}{y+x}&=\frac{y'\mathbf{e}+x'\mathbf{f}}{y'+x'}=\mathbf{q} \\ \frac{z\mathbf{a}+x\mathbf{c}}{z+x}&=\frac{z'\mathbf{d}+x'\mathbf{f}}{z'+x'}=\mathbf{r} \\ \frac{z\mathbf{a}-y\mathbf{b}}{z-y}&=\frac{z'\mathbf{d}-y'\mathbf{e}}{z'-y'}=\mathbf{p} \end{aligned}\right\} \quad \cdots \quad (6)$$

Using (5) we get from (6),

$$y\mathbf{b}+x\mathbf{c}=k(y'\mathbf{e}+x'\mathbf{f}) \text{ i.e., } \frac{y\mathbf{b}-ky'\mathbf{e}}{y-ky'}=\frac{kx'\mathbf{f}-x\mathbf{c}}{kx'-x}$$

and $z\mathbf{a}+x\mathbf{c}=k(z'\mathbf{d}+x'\mathbf{f})$ *i.e.,* $\dfrac{z\mathbf{a}-kz'\mathbf{d}}{z-kz'}=\dfrac{kx'\mathbf{f}-x\mathbf{c}}{kx'-x}$

Thus $\dfrac{y\mathbf{b}-ky'\mathbf{e}}{y-ky'}=\dfrac{kx'\mathbf{f}-x\mathbf{c}}{kx'-x}=\dfrac{z\mathbf{a}-kz'\mathbf{d}}{z-kz'}=\mathbf{h}.$

Hence EB, FC and DA are concurrent at O.

(C) Theorem of Ceva. *If the lines joining the vertices A, B, C of a triangle to a point P, in the plane of the triangle, cut the opposite sides in D, E, F respectively, then the product of the ratios in which these points divide BC, CA, AB is equal to unity and conversely, if the product of the three ratios be unity the three lines AD, BE, CF are concurrent.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{p}$ be position vectors of A, B, C, P with reference to a certain origin. Since A, B, C, P are coplanar, there exists four scalars l, m, n, t such that

$$l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+t\mathbf{p}=\mathbf{0}, \quad \text{where } l+m+n+t=0.$$

They give, $$\frac{m\mathbf{b}+n\mathbf{c}}{m+n}=\frac{l\mathbf{a}+t\mathbf{p}}{l+t}=\mathbf{d}.$$

Thus D whose position vector is $\mathbf{d}$, divides BC in the ratio $n:m$. Similarly we can show that E divides CA in the ratio $l:n$, and F divides AB in the ratio $m:l$.

The product of these three ratios is unity. This proves the theorem.

The Converse is also true.

Let D, E, F be the three points on BC, CA and AB respectively such that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1.$$

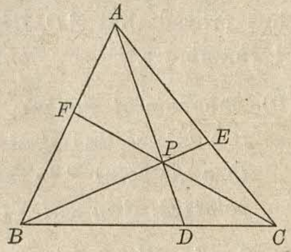


Fig. 2.10. Theorem of Ceva

Assume $BD:DC=n:m$, $CE:EA=l:n$

so that $AF:FB$ should be equal to $m:l$.

Thus, $$\mathbf{d}=\frac{m\mathbf{b}+n\mathbf{c}}{m+n},\quad \mathbf{e}=\frac{l\mathbf{a}+n\mathbf{c}}{l+n},\quad \mathbf{f}=\frac{m\mathbf{b}+l\mathbf{a}}{m+l}.$$

On the join of $\mathbf{a}$ and $\mathbf{d}$ let us take a point P which divides in the ratio $m+n:l$. Its position vector $\mathbf{p}$ is given by

$$\mathbf{p}=\frac{l\mathbf{a}+m\mathbf{b}+n\mathbf{c}}{l+m+n}.$$

By symmetry, we can show that this point also lies on the join of $\mathbf{b}$ and $\mathbf{e}$ and on the join of $\mathbf{c}$ and $\mathbf{f}$. Hence the three lines AD, BE, CF are concurrent at P.

Alternative proof of the converse.

As before, we deduce

$$\mathbf{d}=\frac{m\mathbf{b}+n\mathbf{c}}{m+n}.$$

This implies that **d** is the centroid of $m\mathbf{b}$ and $n\mathbf{c}$. Now suppose P is the centroid of $l\mathbf{a}$, $m\mathbf{b}$ and $n\mathbf{c}$, so that

$$\begin{aligned}(l+m+n)\,\mathbf{p} &= l\mathbf{a}+m\mathbf{b}+n\mathbf{c}\\ &= l\mathbf{a}+(m+n)\,\mathbf{d}.\end{aligned}$$

Hence P lies on AD and similarly on BE and CF. We have thus proved that AD, BE, CF meet at a point P, the centroid of $l\mathbf{a}$, $m\mathbf{b}$, $n\mathbf{c}$.

(D) Theorem of Pascal. *If A, B, C are points on one of the two intersecting straight lines and A', B', C' are on the other, then the point P in which BC' cuts $B'C$, is collinear with the points Q, R in which CA' cuts $C'A$ and AB' cuts $A'B$.*

Proof. We give below the outlines of the proof leaving the details for the students. Suppose H is the point of intersection of the two lines (Fig. 2. 11). Then if H divides BC in the ratio $-b : c$ and $B'C'$ in the ratio $-b' : c'$ we then have

$$\mathbf{h}=\frac{c\mathbf{b}-b\mathbf{c}}{c-b}=\frac{c'\mathbf{b}'-b'\mathbf{c}'}{c'-b'}.$$

Hence, $$\frac{c(c'-b')\mathbf{b}+b'(c-b)\mathbf{c}'}{cc'-bb'}=\frac{c'(c-b)\mathbf{b}'+b(c'-b')\mathbf{c}}{cc'-bb'}=\mathbf{p}.$$

We can construct similar expressions for **q** and **r** and then obtain the linear relation

$$aa'(cc'-bb')\,\mathbf{p}+bb'(aa'-cc')\,\mathbf{q}+cc'(bb'-aa')\mathbf{r}=\mathbf{0},$$

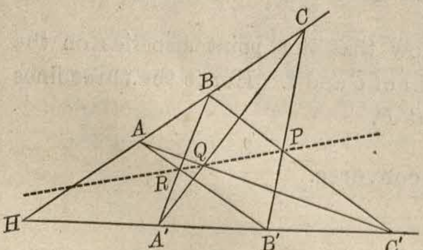


Fig. 2.11. Theorem of Pascal

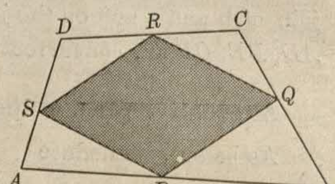


Fig. 2.12. Theorem of Carnot

where the sum of the scalar coefficients of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is zero. Hence the points P, Q, R are collinear.

Otherwise. Let $\mathbf{a} = \alpha\mathbf{e}$, $\mathbf{a}' = \alpha'\mathbf{e}'$ where $\mathbf{e}$, $\mathbf{e}'$ are unit vectors along the lines $H\,ABC$ and $H\,A'B'C'$. Similar expressions for $\mathbf{b}, \mathbf{b}', \mathbf{c}, \mathbf{c}'$. Then obtain $\Sigma\alpha\alpha'(\gamma\gamma' - \beta\beta')\,\mathbf{p} = \mathbf{0}$. Hence etc.

(E) Theorem of Carnot. *The product of the ratios in which a plane PQRS cuts the sides AB, BC, CD, DA of a skew quadrilateral is equal to unity.*

Proof. Take A as origin.

Let $\dfrac{AP}{PB} = k$, $\dfrac{BQ}{QC} = l$, $\dfrac{CR}{RD} = m$, $\dfrac{DS}{SA} = n$.

Then, $\mathbf{p} = \dfrac{k\mathbf{b}}{k+1}$, $\mathbf{q} = \dfrac{l\mathbf{c} + \mathbf{b}}{l+1}$, $\mathbf{r} = \dfrac{m\mathbf{d} + \mathbf{c}}{m+1}$, $\mathbf{s} = \dfrac{\mathbf{d}}{n+1}$.

Now, $(l+1)\,\mathbf{q} - (m+1)\,l\mathbf{r} = \mathbf{b} - lm\mathbf{d} = \dfrac{k+1}{k}\,\mathbf{p} - (n+1)\,lm\mathbf{s}$

$$\text{hence,}\quad \frac{k+1}{k}\,\mathbf{p} - (l+1)\,\mathbf{q} + (m+1)\,l\mathbf{r} - (n+1)\,lm\mathbf{s} = \mathbf{0} \ldots \quad (1)$$

But P, Q, R, S are coplanar and their position vectors are connected by the linear relation (1) and hence the sum of the coefficients should be zero.

$$\text{i.e.,}\quad \frac{k+1}{k} - (l+1) + (lm + l) - (lmn + lm) = 0$$

$$\text{i.e.,}\quad (k+1) - (kl + k) + (klm + kl) - (klmn + klm) = 0$$

$$\text{i.e.,}\quad 1 - klmn = 0 \quad \text{i.e.,}\quad klmn = 1.$$

This proves the theorem.

(F) Theorems on Complete Quadrangles and Complete Quadrilaterals.

DEFINITION 1. A Complete quadrangle (Fig. 2.13), has *four coplanar points* A, B, C, D, no three collinear, and the *six lines* which joins them.

The four points are called the *vertices*, and the six lines (*thick lines of the figure*), *the sides* of the complete quadrangle.

Two sides which do not meet at a vertex are called *opposite sides*. Thus in the figure three pairs of opposite sides (AB, DC) ; (AD, BC) ; (AC, BD) meet at three points P, Q, R respectively ; these points are called *diagonal points* and the triangle PQR formed by the diagonal points is called the *diagonal triangle*.

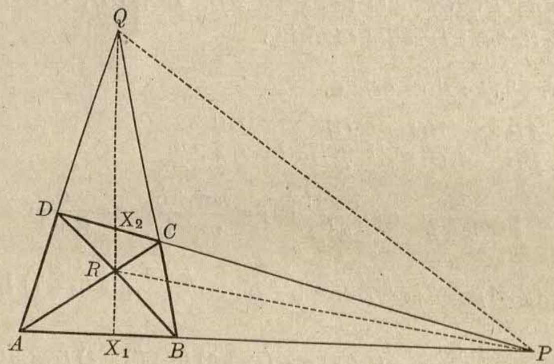


Fig. 2.13. Complete Quadrangle

DEFINITION 2. A complete quadrilateral consists of *four lines*, no three concurrent and the *six points* in which they intersect (Fig. 2.14).

The four lines are called the *sides*, and the six points, the *vertices* of the complete quadrilateral.

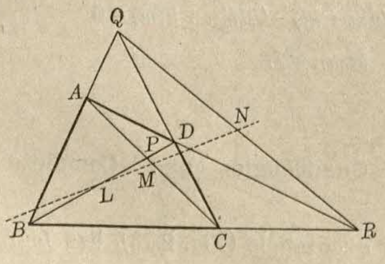


Fig. 2.14. Complete Quadrilateral

Two vertices of a complete quadrilateral which do not lie on a side of the quadrilateral are called *opposite vertices*, and the line joining them is known as a *diagonal*. Thus in the figure the three diagonals are AC, BD, QR.

Note. The geometrical elements—*point* and *line*—are called *dual elements*. The drawing of a line through a point and the marking of a point on a line are known as *dual operations.*

Two figures consisting of points and lines shall be called *dual* if one figure can be obtained from the other by replacing each element in it by the dual element and each operation by the dual operation. Thus the dual of a complete quadrangle is a complete quadrilateral, and vice versa.

On complete quadrangles.

First result. *It is required to find the position vectors of the diagonal points P, Q, R of the complete quadrangle of Fig. 2.13.*

Solution. Since A, B, C, D are coplanar, there exist four scalars x, y, z, t such that

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+t\mathbf{d}=\mathbf{0}, \text{ where } x+y+z+t=0.$$

Now deduce,

$$\left.\begin{aligned} \frac{x\mathbf{a}+y\mathbf{b}}{x+y}&=\frac{z\mathbf{c}+t\mathbf{d}}{z+t}=\mathbf{p} \\ \frac{x\mathbf{a}+t\mathbf{d}}{x+t}&=\frac{y\mathbf{b}+z\mathbf{c}}{y+z}=\mathbf{q} \\ \frac{x\mathbf{a}+z\mathbf{c}}{x+z}&=\frac{y\mathbf{b}+t\mathbf{d}}{y+t}=\mathbf{r} \end{aligned}\right\} \quad \cdots \quad \cdots \quad (1)$$

Second result. *Suppose the line joining Q and R intersects AB* at X_1 and DC at X_2. *It is required to find the position vectors of* X_1 *and* X_2.

Solution. From the relations (1), we write

$$(x+z)\,\mathbf{r}=x\mathbf{a}+z\mathbf{c}$$

$$(y+z)\,\mathbf{q}=y\mathbf{b}+z\mathbf{c}$$

whence it follows,

$$\left.\begin{aligned} \frac{(x+z)\mathbf{r}-(y+z)\mathbf{q}}{(x+z)-(y+z)}&=\frac{x\mathbf{a}-y\mathbf{b}}{x-y}=\mathbf{x}_1 \\ \text{Similarly,}\quad \frac{(x+z)\mathbf{r}-(x+t)\mathbf{q}}{(x+z)-(x+t)}&=\frac{z\mathbf{c}-t\mathbf{d}}{z-t}=\mathbf{x}_2 \end{aligned}\right\} \quad \cdots \quad (2)$$

Third result. *It is required to prove that* X_1, P *are harmonic conjugates to* A, B ; X_2, P *are harmonic conjugates to* D, C *and* X_1, X_2 *are harmonic conjugates to* Q, R.

Proof. We observe from (2) and (1) that

$$(x-y)\,\mathbf{x}_1 = x\mathbf{a} - y\mathbf{b},$$
$$(x+y)\,\mathbf{p} = x\mathbf{a} + y\mathbf{b}.$$

This implies that X_1 divides AB in the ratio $-y : x$ and that P divides AB in the ratio $y : x$. Hence X_1, P are harmonic conjugates to A, B. Similarly other results may be proved.

Further we may prove the following :

Suppose AC *cuts* PQ *at* Z_1 ; BD *cuts* PQ *at* Z_2. *Then* Z_1, R *are harmonic conjugates to* A, C ; *also* Z_2, R *are harmonic conjugates to* B, D. *Further* Z_1, Z_2 *are harmonic conjugates to* P, Q.

The statements of similar results for the sides AD, BC may be constructed without difficulty.

Conclusion 1. *Two vertices of a complete quadrangle are separated harmonically by the diagonal point on their side and the point of intersection of their side with the line joining the other two diagonal points.*

Conclusion 2. *Two diagonal points of a complete quadrangle are separated harmonically by the points in which their line is intersected by the sides passing through the third diagonal point.*

Fourth result. *The mid-points of* AC, BD *and* PQ *of the complete quadrangle* $ABCD$ (Fig. 2.13) *are collinear.*

Proof. From the results

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = \mathbf{0},\ x+y+z+t=0\ ;$$
$$(x+y)\,\mathbf{p} = x\mathbf{a} + y\mathbf{b}, \quad (y+z)\,\mathbf{q} = y\mathbf{b} + z\mathbf{c}\ ;$$

we deduce

$$\begin{aligned}(x+y)(y+z)(\mathbf{p}+\mathbf{q}) &= (y+z)(x\mathbf{a}+y\mathbf{b})+(x+y)(y\mathbf{b}+z\mathbf{c})\\ &= xz(\mathbf{a}+\mathbf{c})+y(2y+x+z)\mathbf{b}+y(x\mathbf{a}+z\mathbf{c})\\ &= xz(\mathbf{a}+\mathbf{c})+y(y-t)\mathbf{b}+y(-y\mathbf{b}-t\mathbf{d})\\ &= xz(\mathbf{a}+\mathbf{c})-yt(\mathbf{b}+\mathbf{d}).\end{aligned}$$

$\therefore$ the mid-points of AC, BD, PQ will satisfy a linear relation where the sum of the scalar coefficients is zero. This proves their collinearity.

On Complete Quadrilaterals.

First result. *The mid-points of the diagonals of a complete quadrilateral are collinear.*

Proof. The points Q, B, C, P (Fig. 2.14) are coplanar and hence there exist four scalars x, y, z, t such that

$$x\mathbf{q}+y\mathbf{b}+z\mathbf{c}+t\mathbf{p}=\mathbf{0},\ x+y+z+t=0$$

whence,

$$\frac{x\mathbf{q}+z\mathbf{c}}{x+z}=\frac{y\mathbf{b}+t\mathbf{p}}{y+t}=\mathbf{d},$$

$$\frac{x\mathbf{q}+y\mathbf{b}}{x+y}=\frac{z\mathbf{c}+t\mathbf{p}}{z+t}=\mathbf{a}.$$

We further deduce

$$\frac{(x+z)\,\mathbf{d}-(x+y)\mathbf{a}}{z-y}=\frac{z\mathbf{c}-y\mathbf{b}}{z-y}=\mathbf{r}.$$

The mid-points L, M, N of BD, AC and QR respectively are then given by

$$\mathbf{l}=\frac{\mathbf{b}+\mathbf{d}}{2}=\frac{\mathbf{b}}{2}+\frac{x\mathbf{q}+z\mathbf{c}}{2(x+z)}=\frac{x\mathbf{q}+(x+z)\mathbf{b}+z\mathbf{c}}{2(x+z)}$$

$$\mathbf{m}=\frac{\mathbf{a}+\mathbf{c}}{2}=\frac{x\mathbf{q}+y\mathbf{b}}{2(x+y)}+\frac{\mathbf{c}}{2}=\frac{x\mathbf{q}+y\mathbf{b}+(x+y)\mathbf{c}}{2(x+y)}$$

$$\mathbf{n}=\frac{\mathbf{q}+\mathbf{r}}{2}=\frac{\mathbf{q}}{2}+\frac{z\mathbf{c}-y\mathbf{b}}{2(z-y)}=\frac{(z-y)\mathbf{q}-y\mathbf{b}+z\mathbf{c}}{2(z-y)}.$$

It is now easy to see that

$$y(x+z)\mathbf{l} - z(x+y)\mathbf{m} + x(z-y)\mathbf{n} = \mathbf{0},$$

where the sum of the scalar coefficients is zero ; this proves the collinearity of L, M, N.

Second result. *Each diagonal of a complete quadrilateral is cut harmonically by the other two.*

Proof. The points B, P, D have the position vectors

$$\mathbf{b}, \frac{x\mathbf{q} + y\mathbf{b} + z\mathbf{c}}{x+y+z}, \frac{x\mathbf{q} + z\mathbf{c}}{x+z}.$$

$\therefore$ P divides BD in the ratio $(x+z) : y$.

The point which divides BD in the ratio $(x+z) : -y$ has the position vector $\dfrac{-y\mathbf{b} + x\mathbf{q} + z\mathbf{c}}{x+z-y}$ which is same as

$$\frac{\dfrac{z\mathbf{c} - y\mathbf{b}}{z-y}(z-y) + x\mathbf{q}}{x+(z-y)} = \frac{(z-y)\mathbf{r} + x\mathbf{q}}{x+(z-y)}.$$

Thus the point lies on QR. Hence etc.

Examples. II(C)

1. Prove that the joins of the mid-points of opposite edges of a tetrahedron intersect and bisect each other. Also prove that the lines joining the vertices of a tetrahedron to the centroids of the areas of the opposite faces are concurrent.

2. Show that if a tetrahedron is cut by a plane parallel to two opposite edges then the section is a parallelogram.

3. Prove that the six planes which contain one edge and bisect the opposite edge of a tetrahedron meet at a point.

4. Verify that the mid-points of the six edges of a cube which do not meet a particular diagonal are coplanar.

5. The straight lines through the mid-points of three coplanar edges of a tetrahedron, each parallel to the line joining a fixed point O to the mid-point of the opposite edge, are concurrent at P, such that OP is bisected by the centroid of the tetrahedron.

6. The bisectors of the opposite sides of a quadrangle $ABCD$ (Fig. 2.13) meet in the centroid of its vertices and are all bisected there.

7. A line cuts the sides of a polygon $P_1, P_2, \ldots P_n$ in n distinct points $R_1, R_2, \ldots R_n$. Show that the product of the ratios in which R_1 divides P_1P_2, R_2 divides $P_2P_3, \ldots$, R_n divides P_nP_1 is $(-1)^n$.

8. If any point O be joined to the vertices of a tetrahedron $ABCD$ and AO, BO, CO, DO are produced to cut the planes of the opposite faces in P, Q, R, S respectively, then

$$\Sigma OP/AP = 1.$$

2·5. Elementary Applications in Mechanics.

(A) Concurrent Forces. A force has magnitude, direction as well as a *definite line of action.* Effect of two forces acting on a rigid body along two *different* parallel lines will be different even if they may have same magnitude and direction. Thus a force cannot be completely represented by a single vector. To represent a force we need the concept of a *Line vector*—a vector which is restricted to lie in a definite line. Two line vectors are equal when they have the same length, same support and same sense. A force acting on a body can be represented by a line vector.

We shall, in this article, discuss only the system of concurrent forces ; more precisely, we shall confine ourselves to forces acting on a single particle.

The joint action of two concurrent forces produces the same dynamical effect as that of a single force, equivalent to their

vector sum and acts through the point of concurrence. Again the joint action of several forces represented by vectors

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots\ldots \mathbf{F}_n$$

acting at a point P, will be the same as the action of a single force represented by a vector $\mathbf{R}$ where

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \Sigma \mathbf{F}$$

acting through the same point P ; $\mathbf{R}$ is called the *resultant* of the system of forces. In order to obtain $\mathbf{R}$ we construct the *vector*

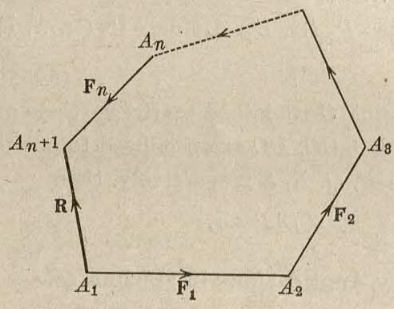


Fig. 2.15. Resultant of several forces

polygon i.e., a polygon whose lengths and directions of the sides being those of vectors

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots \mathbf{F}_n.$$

Thus, suppose in the polygon $A_1 A_2 A_3, \ldots A_n, A_{n+1}$,

$$\overrightarrow{A_1 A_2} = \mathbf{F}_1, \ \overrightarrow{A_2 A_3} = \mathbf{F}_2, \ldots \overrightarrow{A_n A_{n+1}} = \mathbf{F}_n.$$

The closing side of the polygon $A_1 A_{n+1}$ (Fig. 2.15) taken in the opposite order in which the forces have been drawn represents $\mathbf{R}$. In general, the forces

$$\mathbf{F}_1, \mathbf{F}_2, \ldots\ldots \mathbf{F}_n$$

will not form a closed polygon. But in case the polygon is closed, the resultant $\mathbf{R} = \mathbf{0}$ and we say that the *system of forces is*

in equilibrium. Further the vector polygon is not necessarily a plane unless the forces are coplanar.

Again as a particular case if three forces acting at a point are in equilibrium then the closed vector polygon is a triangle **(Triangle of forces).** The forces will then be obviously coplanar and the *length* of each force vector is proportional to the sine of the angle between the other two **(Lami's Theorem).**

Suppose there are n concurrent forces and with respect to an assigned origin O let OA_1, $OA_2 \ldots\ldots OA_n$ represent these forces. Then

$$\mathbf{R} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = n\,\overrightarrow{OG},$$

where G is the centroid of the points $A_1, A_2, \ldots A_n$. The forces will be in equilibrium if G coincides with O.

(B) Displacement. The displacement from A to B is the vector $\overrightarrow{AB}$. Two successive displacements, one from A to B

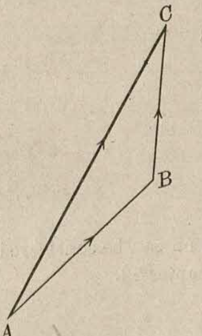


Fig. 2.16. Displacement

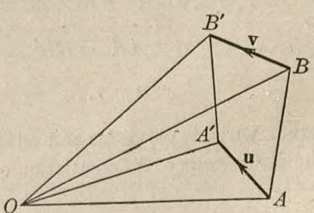


Fig. 2.17. Relative velocity

followed by another from B to C represented by $\overrightarrow{AB}$, $\overrightarrow{BC}$, is the displacement $\overrightarrow{AC}$ (vector addition) ; see Fig. 2.16.

(C) Relative Position. When two points A and B both move, the *relative position* at any instant of the point A with respect to B is the vector $\overrightarrow{BA}$ *i.e.*, position vector of A with respect to B as origin.

(D) Relative Velocity. Relative velocity of a moving point A with respect to another moving point B is the instantaneous rate of change of position of A relative to B. If, however, two points A and B both move at a uniform rate then the relative velocity of A with respect to B is the change in their relative positions per unit of time.

Theorem. *The relative velocity of A with respect to B is the vector difference of velocities* $\mathbf{u}$ *and* $\mathbf{v}$ *of A and B relative to a fixed point O ; A and B being supposed to move at uniform rates* (Fig. 2.17).

Proof. Suppose A and B move to A', B' in one sec. Then, by definition, relative velocity of A with respect to B

$$
\begin{aligned}
&= \overrightarrow{B'A'} - \overrightarrow{BA}. \\
&= (\overrightarrow{OA'} - \overrightarrow{OB'}) - (\overrightarrow{OA} - \overrightarrow{OB}) \\
&= (\overrightarrow{OA'} - \overrightarrow{OA}) - (\overrightarrow{OB'} - \overrightarrow{OB}) \\
&= \overrightarrow{AA'} - \overrightarrow{BB'} \\
&= \mathbf{u} - \mathbf{v}.
\end{aligned}
$$

Note. Variable velocities and relative acceleration can be considered only after the discussions of differentiation of vectors (chapter 5).

Examples. II(D)

1. If the resultant of two forces acting on a particle be at right angles to one of them and its magnitude be one-third of the other, show that the ratio of the larger force to the smaller is $3 : 2\sqrt{2}$.

2. **R** is the resultant of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$; $\mathbf{F}_1$ acts horizontally and **R** vertically but $\mathbf{F}_2$ is inclined at 60° to the vertical. Find the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ in terms of the magnitude of **R**.

3. If the resultant of two forces be equal in magnitude to one of the components and perpendicular to its direction, find the other component.

4. Apply the principle of vectors to prove that three concurrent forces represented in magnitude and direction by the medians of a triangle are in equilibrium.

5. If two concurrent forces are given by $n.\overrightarrow{OA}$ and $m.\overrightarrow{OB}$ respectively, show that their resultant is given by $(m+n)\ \overrightarrow{OR}$ where R divides AB such that $n\overrightarrow{AR}=m\overrightarrow{RB}$.

6. $ABCD$ is a quadrilateral ; forces $\overrightarrow{AB}$ and $\overrightarrow{AD}$ act at A and $\overrightarrow{CD}$ and $\overrightarrow{CB}$ act at C. Prove that their resultant is $4\overrightarrow{LM}$ where L and M are the middle points of AC, BD respectively.

7. $ABCDEF$ is a regular hexagon ; five forces $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ act at A. Prove that their resultant is $6\overrightarrow{AO}$ where O is the centroid of the hexagon.

8. A, B, C are fixed points and P is a variable point such that the resultant of forces at P represented by $\overrightarrow{PA}$, $\overrightarrow{PB}$ always passes through C. Prove that the locus of P is a straight line joining C to the mid-point of AB.

9. Forces P, Q, R act at O and are in equilibrium. If any transversal cuts their lines at A, B, C respectively, show that $\dfrac{P}{OA}+\dfrac{Q}{OB}+\dfrac{R}{OC}=0$. Generalise this result.

10. D, E, F are the mid-points of BC, CA, AB respectively of the $\triangle ABC$. Show that the system of concurrent forces $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ is equivalent to the system represented by $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$ where O is any point in the plane of the triangle.

11. ABC is a triangle and P is any point on BC. If the resultant of forces represented by $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$ be $\overrightarrow{PQ}$, the locus of Q is a straight line parallel to BC.

12. A man travelling east at 4 miles per hour finds that the wind seems to blow directly from the north. On doubling his speed he finds that it appears to come from north-east. Find the velocity of the wind.

13. A ship whose head is pointing due south across a current running due west; at the end of two hours it is found that the ship has gone 36 miles in a direction 15° west of south. Find the velocities of the ship and the current, graphically and analytically.

14. (*a*) Two particles A and B are moving with velocities u and v along two lines inclined at an angle α. Find the magnitude and direction of the relative velocity of B with respect to A in terms of u, v, α.

(*b*) Two particles move with speeds v and $2v$ respectively on the circumference of a circle. In what positions, is their relative velocity greatest and least and what values has it in those positions ?

15. A particle placed at O is acted on by forces represented by the lines OA_1, OA_2,......, OA_n where A_1, A_2,......, A_n are fixed points. Where must O be placed that the magnitude of the resultant force may be a constant ?

16. A point describes a circle uniformly in the **i**-**j** plane taking 12 seconds for one complete revolution. If its initial

position vector relative to the centre be $\mathbf{i}$ and the rotation is from $\mathbf{i}$ to $\mathbf{j}$, find the position vectors at the end of 1, 3, 5, 7 seconds ; also at the end of $1\frac{1}{2}$ and $4\frac{1}{2}$ seconds. Find also the velocity vectors of the moving point at the end of $1\frac{1}{2}$, 3 and 7 seconds.

17. The velocity of A relative to B is $3\mathbf{i}+4\mathbf{j}$ and that of B relative to C is $\mathbf{i}-3\mathbf{j}$. Find the velocity of A relative to C ; $\mathbf{i}$ and $\mathbf{j}$ represent velocity of one mile per hour along east and north respectively.

18. Two particles A and B are, at some instant, 15 miles apart. Both start at the same moment. A moves towards B with a uniform velocity of 5 m.p.h. and B moves perpendicular to AB at 3·75 m.p.h. Find their relative velocity and the time when they are nearest to each other and also calculate the shortest distance between them.

19. Prove that the magnitude of the resultant $\mathbf{R}$ of any (finite) number of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \cdots\cdots$, is given by

$$\mathbf{R}^2=\Sigma\mathbf{P}^2+2\Sigma\mathbf{P}_r\mathbf{P}_s\cos\,(\widehat{\mathbf{P}_r, \mathbf{P}_s}).$$

20. $ABCDEFA$ is a regular hexagon. Show that the resultant of the forces represented by AB, $2AC$, $3AD$, $4AE$, $5AF$ may also be represented by a vector of magnitude $\sqrt{351}\,AB$ and find its direction.

Hints and Answers

1. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two two forces and let $\mathbf{R}$ be their resultant perpendicular to $\mathbf{F}_1$. Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors along the two perpendicular directions $\mathbf{F}_1$ and $\mathbf{R}$ respectively. If $|\mathbf{F}_1|=a$ then $\mathbf{F}_1=a\mathbf{i}$. Since $\mathbf{F}_2$ lies in the plane of $\mathbf{i}$ and $\mathbf{j}$ we may take $\mathbf{F}_2=x\mathbf{i}+y\mathbf{j}$ and hence $|\mathbf{F}_2|=\sqrt{x^2+y^2}$. Therefore, by the problem, $\mathbf{R}=\frac{1}{3}\sqrt{x^2+y^2}\,\mathbf{j}$. Since $\mathbf{F}_1+\mathbf{F}_2=\mathbf{R}$, we have $a\mathbf{i}+(x\mathbf{i}+y\mathbf{j})=\frac{1}{3}\sqrt{x^2+y^2}\,\mathbf{j}$, or $(a+x,\ y)=(0,\ \frac{1}{3}\sqrt{x^2+y^2})$; hence $a+x=0$, $y=\frac{1}{3}\sqrt{x^2+y^2}$. This gives $x=-a$, $y=a/2\sqrt{2}$.

$$\therefore\quad |\mathbf{F}_2|:|\mathbf{F}_1|=(\sqrt{a^2+(1/2\sqrt{2})^2a^2}):a=3:2\sqrt{2}.$$

2. $\mathbf{F}_1 = a\mathbf{i}$; $\mathbf{F}_2 = -b\sin 60^\circ\,\mathbf{i} + b\cos 60^\circ\,\mathbf{j}$ where $|\mathbf{F}_1| = a$ and $|\mathbf{F}_2| = b$, If $|\mathbf{R}| = c$, then $\mathbf{R} = c\mathbf{j}$. As before, $(a, 0) + (-b\sqrt{3}/2,\ b/2) = (0, c)$. Now obtain a and b in terms of c.

3. The other component is $\sqrt{2}a$, if the given component has a magnitude a. The direction is inclined at an angle 135° with it.

5. Use Section ratio.

6. $\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AM}$; $\overrightarrow{CD} + \overrightarrow{CB} = 2\overrightarrow{CM}$;

$$2(\overrightarrow{AM} + \overrightarrow{CM}) = 2(\overrightarrow{AL} + \overrightarrow{LM} + \overrightarrow{CL} + \overrightarrow{LM}) = 4\overrightarrow{LM}. \quad (\because\ \overrightarrow{AL} + \overrightarrow{CL} = \mathbf{0})$$

8. Suppose the position vectors of A, B, C, P are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{p}$ with reference to a certain origin O. Then

$$\overrightarrow{PA} + \overrightarrow{PB} = \mathbf{a} - \mathbf{p} + \mathbf{b} - \mathbf{p} = 2(\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}) - 2\mathbf{p} = 2(\mathbf{d} - \mathbf{p}) = 2\overrightarrow{PD},$$

where $\mathbf{d}$ is the position vector of D, the mid-point of AB. Thus the resultant of $\overrightarrow{PA}$ and $\overrightarrow{PB}$ always passes through C and D *i.e.*, P always lies on CD.

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Now the force vector $\mathbf{P} = x\mathbf{a}$, where $x = \dfrac{|\mathbf{P}|}{OA} = \dfrac{P}{OA}$. Similarly, $\mathbf{Q} = y\mathbf{b}$, $\mathbf{R} = z\mathbf{c}$ where $y = \dfrac{Q}{OB}$, $z = \dfrac{R}{OC}$. But $\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{0}$, *i.e.*, $\dfrac{P}{OA}\mathbf{a} + \dfrac{Q}{OB}\mathbf{b} + \dfrac{R}{OC}\mathbf{c} = \mathbf{0}$. Since A, B, C are collinear, the sum of the scalar coefficients in this linear relation is zero. Hence etc.

Generalisation. $\Sigma \dfrac{P}{OA} = 0$. Proceed similarly.

11. Take A as origin. Suppose D is the mid-point of BC. With usual notations for position vectors, $\Sigma\overrightarrow{PA} = -\mathbf{p} + (\mathbf{b} - \mathbf{p}) + (\mathbf{c} - \mathbf{p}) = (\mathbf{b} + \mathbf{c}) - 3\mathbf{p}$

$$= \mathbf{q} - \mathbf{p} = \overrightarrow{PQ},$$

where $\mathbf{q} = \mathbf{b} + \mathbf{c} - 2\mathbf{p} = 2(\mathbf{d} - \mathbf{p})$.

Thus $\mathbf{q} = \overrightarrow{AQ} = 2\overrightarrow{PD}$; hence $\overrightarrow{AQ} \parallel \overrightarrow{PD}$ and consequently $\overrightarrow{AQ} \parallel \overrightarrow{BC}$.

12. Let $\mathbf{i}$ and $\mathbf{j}$ represent the unit velocity-vectors along *East* and *North* respectively. In the first case the velocity of the man is $4\mathbf{i}$. Let the true velocity of the wind be $x\mathbf{i} + y\mathbf{j}$. Therefore, the velocity of the wind relative to the man is $(x\mathbf{i} + y\mathbf{j}) - 4\mathbf{i} = (x - 4)\mathbf{i} + y\mathbf{j}$. But it is given that this velocity is directed from the north and hence $= m\mathbf{j}$ (say). This will give $x - 4 = 0$ or $x = 4$.

In the second case, when the man doubles his speed the velocity of the man is $8\mathbf{i}$ and hence relative velocity is $(x\mathbf{i}+y\mathbf{j})-8\mathbf{i}$. But this comes from North-East and hence this vector will be parallel to $k\cos 45^\circ\mathbf{i}+k\sin 45^\circ\mathbf{j}$ $(k/\sqrt{2})(\mathbf{i}+\mathbf{j})$. Hence $x-8=k/\sqrt{2}$ and $y=k/\sqrt{2}$. This gives $k=-4\sqrt{2}$ and hence $y=-4$. Therefore the true velocity of the wind is is $\overrightarrow{OP}=4\mathbf{i}-4\mathbf{j}$; its magnitude $=4\sqrt{2}$ m. p. h. and direction is from North-West to South-East.

13. $18\cos 15^\circ=17{\cdot}4$ m. p. h. ; $18\sin 15^\circ=4{\cdot}7$ m. p. h.

14. (*a*) $(u^2+v^2-2uv\cos\alpha)^{\frac{1}{2}}$ at an angle $\tan^{-1}\dfrac{v\sin\alpha}{v\cos\alpha-u}$ with the direction of P.

(*b*) $3v$ or v according as the points are moving in the opposite or in the same direction when they are at the ends of a diameter.

15. O may be placed anywhere on a sphere of radius R/n, R being the magnitude of the resultant ; the centre of the sphere will be at the centroid of the given points.

16. $\frac{1}{2}(\sqrt{3}, 1)$; $(0, 1)$; $\frac{1}{2}(-\sqrt{3}, 1)$; $-\frac{1}{2}(\sqrt{3}, 1)$; $\frac{1}{\sqrt{2}}(1, 1)$; $\frac{1}{\sqrt{2}}(-1, 1)\cdot\frac{\pi}{6\sqrt{2}}(-1, 1)$, $-\frac{\pi}{6}(1, 0)$, $\frac{\pi}{12}(1, -\sqrt{3})$.

17. $\sqrt{17}$ m. p. h. at an angle $\tan^{-1}\frac{1}{4}$ North of East.

18. $6{\cdot}25$ m. p. h. ; $1{\cdot}92$ hours ; 9 miles.

3

Product of Vectors

3·1. Introduction.

We have so far defined addition of two or more vectors, difference of two vectors and multiplication of a vector by a number (scalar). In this chapter we shall define two operations between vectors known as their *products*. The ways in which two or more vectors enter into combination in Geometry, Mechanics or in other branches of applied sciences lead us to define two *different kinds* of products. One of these products will yield a pure number (scalar) and hence called *Scalar product* (also known as *Inner product* or *Dot product*). The other definition of product gives a well-defined vector and hence called *Vector product* (also known as *Outer product* or *Cross product*). It is natural that they should be distinguished by difference in notations. Scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$ (placing a dot between two factors and hence the name *dot product*) while their vector product is represented by $\mathbf{a} \times \mathbf{b}$ (placing a cross in-between and hence the name *cross product*). Their definitions will show that both the products involve the product of the lengths of two factors and each follows the *distributive law*.

3·2. Scalar product : Definition from Geometric stand-point.

The definition of scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ may be given from two different stand-points—one *geometric*, the other purely *algebraic*.

DEFINITION. (*Geometric Stand-point*). The Scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$, written as $\mathbf{a} \cdot \mathbf{b}$, is defined as the product of their lengths $|\mathbf{a}|$ and $|\mathbf{b}|$ and the cosine of their included angle.

Thus

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta,$$

where θ is customarily taken as the smallest angle, taken positive, between **a** and **b**.

Note. **a** . **b** is a number ; it will be also called *dot product* of two vectors **a** and **b**.

Motivations.

The above definition may appear rather arbitrary to one who is unfamiliar with its application. But the reader must be familiar with the definition of *work done by a force*. Thus the work done by a force vector **F** causing a displacement **s** is the product of the magnitude of **s** and the component of **F** in the direction of **s** (or the product of the magnitude of **F** and the component of **s** in the direction of **F**). This combination is a *product* and the result is a scalar since it involves the multiplication of two lengths. Many such combinations arise in applications whence we have suggested the above definition.

3·21. Properties of Scalar products.

From the definition given above we may easily deduce the following properties :

(i) **Commutative Law :** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

For, by definition, $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}| \cos \theta$, where θ is the smallest angle between **b** and **a** (this angle is same as the smallest angle between **a** and **b**).

Thus *in a scalar product the order of the factors may be reversed without altering the value of the product.*

(ii) *The scalar product of two proper vectors* **a** *and* **b** *is positive, zero or negative, according as the angle* θ *between* **a** *and* **b** *is acute, right or obtuse.*

For, $\cos \theta >, =$ or < 0 according as $\theta <, =$ or $> 90°$.

(iii) Vanishing of a . b.

If either **a** or **b** is a zero vector then $\mathbf{a}\cdot\mathbf{b}=0$. If both **a** and **b** are proper vectors then $\mathbf{a}\cdot\mathbf{b}=0$ if they are at right angles.

Thus $\mathbf{a}\cdot\mathbf{b}=0$ does not always imply that either **a** or **b** is a zero vector ; it may be that neither **a** nor **b** is a zero vector but **a** is perpendicular to **b**.

In fact, *for two proper vectors* **a** *and* **b**, *the condition* $\mathbf{a}\cdot\mathbf{b}=0$ *is the condition of perpendicularity of two vectors.*

(iv) For two parallel vectors **a** and **b**, we have $\theta=0$ and hence $\cos\theta=1$.

Thus for two parallel vectors **a** and **b** we have $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|$.

For two equal vectors **a** and **a**, we agree to write

$$\mathbf{a}.\ \mathbf{a}=\mathbf{a}^2.$$

Clearly, then $\mathbf{a}^2=|\mathbf{a}|\,|\mathbf{a}|=|\mathbf{a}|^2$.

i.e., scalar product of a vector by itself is the square of its length.

Note that $(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=|\mathbf{a}+\mathbf{b}|^2$,

$$(\mathbf{a}-\mathbf{b})^2=(\mathbf{a}-\mathbf{b}).(\mathbf{a}-\mathbf{b})=|\mathbf{a}-\mathbf{b}|^2\ ;\ \text{etc.}$$

(v) When **a** and **b** have opposite directions,

$$\mathbf{a}\cdot\mathbf{b}=-|\mathbf{a}|\,|\mathbf{b}|.$$

For, then $\theta=180^\circ$ and $\cos\theta=-1$.

(vi) Scalar Product of two unit vectors a and b.

If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are the unit vectors along **a** and **b** respectively then $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\cos\theta$.

i.e., the scalar product of two unit vectors gives the cosine of the angle between them.

(vii) For the unit vectors **i**, **j**, **k** along three mutually perpendicular directions we have evidently

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1,$$

and $\quad \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0\ ;\ \mathbf{j}\cdot\mathbf{i}=\mathbf{k}\cdot\mathbf{j}=\mathbf{i}\cdot\mathbf{k}=0.$

(viii) Associativity with Scalars : $(m\mathbf{a}).\mathbf{b} = \mathbf{a}.(m\mathbf{b}) = m(\mathbf{a}.\mathbf{b})$.

i.e. if either factor be multiplied by a number, then the scalar product is multiplied by that number.

For, if the angle between **a** and **b** be θ then that between $m\mathbf{a}$ and **b** will be θ or $\pi-\theta$ according as m is positive or negative. Accordingly,

Case 1. When m is positive,

$$(m\mathbf{a}).\mathbf{b} = |m\mathbf{a}|\,|\mathbf{b}| \cos\theta = m|\mathbf{a}|\,|\mathbf{b}| \cos\theta = m(\mathbf{a}.\mathbf{b})$$

and $\quad \mathbf{a}.(m\mathbf{b}) = |\mathbf{a}|\,|m\mathbf{b}| \cos\theta = m|\mathbf{a}|\,|\mathbf{b}| \cos\theta = m(\mathbf{a}.\mathbf{b})$.

Case 2. When m is negative,

$$(m\mathbf{a}).\mathbf{b} = |m\mathbf{a}|\,|\mathbf{b}| \cos(\pi-\theta) = -m|\mathbf{a}|\,|\mathbf{b}| \cos(\pi-\theta)$$
$$= m|\mathbf{a}|\,|\mathbf{b}| \cos\theta = m(\mathbf{a}.\mathbf{b})$$

Also, we have $\mathbf{a}.(m\mathbf{b}) = m(\mathbf{a}.\mathbf{b})$.

More generally, we may prove

$$(m\mathbf{a}).(n\mathbf{b}) = mn(\mathbf{a}.\mathbf{b}) = \mathbf{a}.(mn\mathbf{b}) = m\{(n\mathbf{a}).\mathbf{b}\},$$

where m and n are two numbers, positive or negative.

We leave out the proof of this general case for the students.

Note that $(\mathbf{a}.\mathbf{b}).\mathbf{c}$ is not defined (the dot is only defined between *two vectors*), the associative law between three vectors need not be considered. We, however, point out that $(\mathbf{a}.\mathbf{b})\mathbf{c} \neq \mathbf{a}(\mathbf{b}.\mathbf{c})$ generally, though both are defined.

(ix) Angle between two proper vectors a and b.

Since, $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, it follows

$$\theta = \cos^{-1}\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

(x) Another statement for the definition of a . b.

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\{|\mathbf{b}|\cos\theta\} = |\mathbf{a}|\{\text{Projection of } \mathbf{b} \text{ along } \mathbf{a}\},$$

or, $\quad \mathbf{a}.\mathbf{b} = |\mathbf{b}|\{|\mathbf{a}|\cos\theta\} = |\mathbf{b}|\{\text{Projection of } \mathbf{a} \text{ along } \mathbf{b}\}$.

(xi) Distributive Law. For any three vectors **a, b, c**

$$\mathbf{a}.(\mathbf{b}+\mathbf{c})=\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c}$$

i.e., scalar product is distributive with respect to addition.

Proof. We refer to Fig. 3.1. Let $\overrightarrow{AB}=\mathbf{b}$, $\overrightarrow{BC}=\mathbf{c}$, so that

$$\overrightarrow{AC}=\mathbf{b}+\mathbf{c}.$$

Let L be a directed line which is the line of support of **a** ; the unit vector along L will be denoted by **e**. The projection of **b** on L is some numerical multiple of **e** ; this numerical multiple is called the *component* of **b** along L. Thus, we have the following relation between Projection and Component :

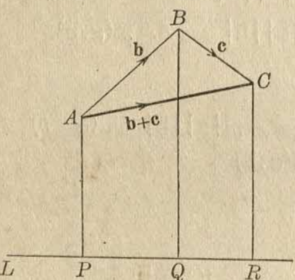


Fig. 3.1. Distributive Law (Scalar product)

$$\text{Proj. } \mathbf{b} \text{ (along } L)$$
$$=\{\text{Comp. } \mathbf{b} \text{ (along } L)\}\mathbf{e}.$$

But we compute Comp. **b** (along L) by $|\mathbf{b}|\cos(\widehat{\mathbf{e}, \mathbf{b}})$

and since $|\mathbf{e}|=1$, we have Comp. **b** (along L) $=\mathbf{e}.\mathbf{b}$ (1)

Now Fig. 3.1 gives,

$$\text{Proj. } \mathbf{b}+\text{Proj. } \mathbf{c}=\text{Proj. } (\mathbf{b}+\mathbf{c}) \quad \cdots \quad (2)$$

$$\therefore \quad (\text{Comp. } \mathbf{b})\mathbf{e}+(\text{Comp. } \mathbf{c})\mathbf{e}=\{\text{Comp.}(\mathbf{b}+\mathbf{c})\}\mathbf{e}$$

$$\text{or,} \quad \text{Comp. } \mathbf{b}+\text{Comp.}\,\mathbf{c}=\text{Comp. } (\mathbf{b}+\mathbf{c}) \quad \cdots \quad (3)$$

Now since, $\mathbf{a}=|\mathbf{a}|\mathbf{e}$, we may write

$$\begin{aligned}
\mathbf{a}.(\mathbf{b}+\mathbf{c}) &= |\mathbf{a}|\mathbf{e}.(\mathbf{b}+\mathbf{c}) \\
&= |\mathbf{a}|\{\text{Comp. } (\mathbf{b}+\mathbf{c})\}, && \text{using (1)} \\
&= |\mathbf{a}|\{\text{Comp. } \mathbf{b}+\text{Comp. } \mathbf{c}\}, && \text{using (3)} \\
&= |\mathbf{a}|\{\mathbf{e}.\mathbf{b}+\mathbf{e}.\mathbf{c}), && \text{using (1)} \\
&= \mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c} && (\because |\mathbf{a}|\mathbf{e}.\mathbf{b}=\mathbf{a}.\mathbf{b})
\end{aligned}$$

Note that Fig. 3.1 has been drawn in which all the projections are positive but the relation (2) holds even if one or both are negative as can be verified by drawing a suitable diagram.

We can also show that $\mathbf{a}.(\mathbf{b}-\mathbf{c})=\mathbf{a}.\mathbf{b}-\mathbf{a}.\mathbf{c}$.

By repeated application of the distributive law we obtain,

$$\begin{aligned}(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{d})&=\mathbf{p}.(\mathbf{c}+\mathbf{d})=\mathbf{p}.\mathbf{c}+\mathbf{p}.\mathbf{d}, \text{ where } \mathbf{p}=\mathbf{a}+\mathbf{b}\\ &=\mathbf{c}.\mathbf{p}+\mathbf{d}.\mathbf{p}=\mathbf{c}.(\mathbf{a}+\mathbf{b})+\mathbf{d}.(\mathbf{a}+\mathbf{b})\\ &=\mathbf{c}.\mathbf{a}+\mathbf{c}.\mathbf{b}+\mathbf{d}.\mathbf{a}+\mathbf{d}.\mathbf{b}\\ &=\mathbf{a}.\mathbf{c}+\mathbf{b}.\mathbf{c}+\mathbf{a}.\mathbf{d}+\mathbf{b}.\mathbf{d}.\end{aligned}$$

More generally, $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\cdots\cdots+\mathbf{k}).(\mathbf{l}+\mathbf{m}+\mathbf{n}+\cdots\cdots+\mathbf{s})$

$$=\{\mathbf{a}.\mathbf{l}+\mathbf{a}.\mathbf{m}+\cdots\cdots+\mathbf{a}.\mathbf{s}\}+\{\mathbf{b}.\mathbf{l}+\mathbf{b}.\mathbf{m}+\cdots\cdots+\mathbf{b}.\mathbf{s}\}$$
$$+\cdots\cdots+\{\mathbf{k}.\mathbf{l}+\mathbf{k}.\mathbf{m}+\cdots\cdots+\mathbf{k}.\mathbf{s}\}.$$

In particular, $(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a}^2+\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{a}+\mathbf{b}^2$

$$=\mathbf{a}^2+2\mathbf{a}.\mathbf{b}+\mathbf{b}^2.$$

Similarly, $(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2-2\mathbf{a}.\mathbf{b}+\mathbf{b}^2$; $\mathbf{a}^2-\mathbf{b}^2=(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b})$.

Note carefully that all the results occur in the familiar form because of the fact $\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{a}$.

(xii) Scalar product in terms of coördinates of the vectors.

Let **a** and **b** be expressed as linear combinations of three mutually perpendicular unit vectors **i, j, k** (they are supposed to form a right-handed system). Thus

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\quad \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\ ;$$

where (a_1, a_2, a_3), (b_1, b_2, b_3) are the coördinates of **a** and **b**. We make use of the distributive law and obtain

$$\begin{aligned}\mathbf{a}.\mathbf{b}&=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\,(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\\ &=\{a_1\mathbf{i}.b_1\mathbf{i}+a_1\mathbf{i}.b_2\mathbf{j}+a_1\mathbf{i}.b_3\mathbf{k}\}\\ &\quad+\{a_2\mathbf{j}.b_1\mathbf{i}+a_2\mathbf{j}.b_2\mathbf{j}+a_2\mathbf{j}.b_3\mathbf{k}\}\\ &\quad+\{a_3\mathbf{k}.b_1\mathbf{i}+a_3\mathbf{k}.b_2\mathbf{j}+a_3\mathbf{k}.b_3\mathbf{k}\}\end{aligned}$$

$$=\{a_1b_1\mathbf{i}^2+a_1b_2\,\mathbf{i}.\mathbf{j}+a_1b_3\,\mathbf{i}.\mathbf{k}\}$$
$$+\{a_2b_1\,\mathbf{j}.\mathbf{i}+a_2b_2\,\mathbf{j}^2+a_2b_3\,\mathbf{j}.\mathbf{k}\}$$
$$+\{a_3b_1\,\mathbf{k}.\mathbf{i}+a_3b_2\,\mathbf{k}.\mathbf{j}+a_3b_3\,\mathbf{k}^2\}, \quad \text{using (viii)}$$
$$=a_1b_1+a_2b_2+a_3b_3 \quad , \text{using (vii)}.$$

Observations. The beginner should carefully note the above steps. Of course, the final result is easy to remember : *Multiply the corresponding coördinates of* **a** *and* **b** *and then add.* *e.g.*, if $\mathbf{a}=3\mathbf{i}+5\mathbf{j}+7\mathbf{k}$, $\mathbf{b}=4\mathbf{i}+2\mathbf{j}+\mathbf{k}$ then $\mathbf{a}.\mathbf{b}=3\times4+5\times2+7\times1=29$. The reader will now appreciate the significance of the term *inner product* which is often used instead of *scalar product*.

(xiii) If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ then

$$|\mathbf{a}|=\sqrt{(a_1{}^2+a_2{}^2+a_3{}^2)}\ ;\ \ |\mathbf{b}|=\sqrt{(b_1{}^2+b_2{}^2+b_3{}^2)}$$

and hence the relation $\mathbf{a}.\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$ gives

$$\theta=\cos^{-1}\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}=\cos^{-1}\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{(a_1{}^2+a_2{}^2+a_3{}^2)}.\sqrt{(b_1{}^2+b_2{}^2+b_3{}^2)}}$$

which indicates the method of finding the angle between two vectors whose coördinates are known.

3·22. Definition of scalar product from Algebraic standpoint.

DEFINITION. Suppose two vectors **a** and **b** are given by the number triples :

$$\mathbf{a}=(a_1,a_2,a_3)\ ;\ \ \mathbf{b}=(b_1,b_2,b_3).$$

We then define $\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$.

Rule. *Multiply the corresponding elements of two vectors and then add.*

Geometric Interpretation : The geometric interpretation of $\mathbf{a}.\mathbf{b}$ is now readily found by considering the triangle OAB formed by vectors **a**, **b** and $\mathbf{a}-\mathbf{b}$ (Fig. 3.2).

Let θ be the smallest angle between **a** and **b**, then from properties of a triangle it follows,

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\,|\mathbf{b}|\cos\theta \qquad \dots \quad (1)$$

But since $\mathbf{a}=(a_1, a_2, a_3)$ and $\mathbf{b}=(b_1, b_2, b_3)$, we have

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2\ ;\ |\mathbf{b}|^2 = b_1^2 + b_2^2 + b_3^2.$$

Also $\mathbf{a}-\mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

$\therefore\ |\mathbf{a}-\mathbf{b}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2.$

From (1) it now follows,

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$
$$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2|\mathbf{a}|\,|\mathbf{b}|\cos\theta.$$

i.e., $a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}|\cos\theta.$

i.e., $\mathbf{a}.\mathbf{b} \quad = |\mathbf{a}|\,|\mathbf{b}|\cos\theta.$

Thus the scalar product of **a** and **b** is the product of their lengths and cosine of their included angle. This is our definition in art. 3·2 (from Geometric stand-point) and as such all the properties deduced in art. 3·21, will be true even if we accept this algebraic definition. To illustrate the use of this definition we append below the proofs of Commutative and Distributive Laws for scalar products.

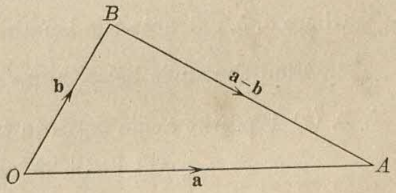


Fig. 3.2. Geometric Interpretation of **a . b**

(i) Commutative Law : If

$$\mathbf{a} = (a_1, a_2, a_3) \text{ and } \mathbf{b} = (b_1, b_2, b_3)$$

then according to our definition,

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3,$$
$$\mathbf{b}.\mathbf{a} = b_1a_1 + b_2a_2 + b_3a_3.$$

It follows $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$. This gives the *Commutative Law.*

(ii) **Distributive Law :** If

$$\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \text{ and } \mathbf{c} = (c_1, c_2, c_3)$$

we now prove

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}.$$

Since $\mathbf{b} + \mathbf{c} = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$

$$\begin{aligned} \therefore \quad \mathbf{a}.(\mathbf{b} + \mathbf{c}) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}. \end{aligned}$$

Hence follows the *Distributive Law.*

Examples. III(A)

1. In the $\triangle ABC$ let $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{c}$. Prove that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

where a, b, c denote the lengths of sides of the triangle.

2. Find the angle between $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (0, 2, 4)$.

3. (*a*) Find the angle between two vectors

$$\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}.$$

(*b*) Find the scalar product of two vectors given by two diagonals of a unit cube. What is the angle between them ?

4. If $\mathbf{a} = (2, 3, 1)$, $\mathbf{b} = (0, 4, 2)$; compute $\mathbf{a}.\mathbf{b}$.

5. If $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$, $\gamma = (c_1, c_2, c_3)$, compute $\beta.\gamma$, $\gamma.\alpha$ and $\alpha.\beta$.

6. Prove that three vectors

$$\alpha = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \beta = \mathbf{i} + \mathbf{j} - 3\mathbf{k}, \gamma = 7\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

are at right angles to each other.

7. If $\mathbf{a}.\mathbf{b}=\mathbf{a}.\mathbf{c}$ then prove that the line joining the end-points of $\mathbf{b}$ and $\mathbf{c}$ (supposed to be drawn from the same origin) is perpendicular to $\mathbf{a}$.

8. A triangle has vertices

$$A\,(1,\ -2,\ 3),\ B\,(2,\ 1,\ -1) \text{ and } C\,(3,\ -1,\ 2).$$

Solve the triangle.

9. $\mathbf{l}.\mathbf{a}=\mathbf{l}.\mathbf{b}=\mathbf{l}.\mathbf{c}=0$ implies that $\mathbf{l}$ is a null-vector, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-coplanar vectors—justify.

10. Show that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

(This point of concurrence is known as the ortho-centre).

11. Show that if $|\mathbf{c}+\mathbf{d}|=|\mathbf{c}-\mathbf{d}|$ then, either vector is perpendicular to the other.

12. Show that the perpendicular bisectors of the sides of a triangle are concurrent.

(This point of concurrence is known as the circum-centre.)

13. Given $\mathbf{a}=\mathbf{i}-2\mathbf{j}$, $\mathbf{b}=\mathbf{j}+\mathbf{k}$, find the component of $\mathbf{a}$ along $\mathbf{b}$.

14. Given $\mathbf{a}=\frac{1}{7}(2,\ 3,\ 6)$, $\mathbf{b}=\frac{1}{7}(3,\ -6,\ 2)$, $\mathbf{c}=\frac{1}{7}(6,\ 2,\ -3)$. Show that $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are each of unit length and are mutually perpendicular.

15. Find the area of the triangle formed by the points

$$A\,(1, 1, 1),\ B\,(1, 2, 3),\ C\,(2, 3, 1).$$

16. If $\mathbf{e}_1$ and $\mathbf{e}_2$ be two unit vectors and θ be the angle between them, show that $2\sin\frac{1}{2}\theta=|\mathbf{e}_1-\mathbf{e}_2|$.

17. In a tetrahedron, if any two pairs of opposite edges are perpendicular then prove that the remaining pair of opposite edges are also perpendicular to each other.

18. In a tetrahedron prove that the sum of the squares on the opposite edges is the same for each pair.

19. Prove, by using the definition of scalar product,

(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

(ii) $a^2 = b^2 + c^2 - 2bc \cos A$.

(iii) $c = b \cos A + a \cos B$

(*Usual notations for sides and angles of a triangle are assumed*).

20. Prove that an angle inscribed in a semi-circle is a right angle.

21. (i) From the identity

$$(\mathbf{a}-\mathbf{b}).(\mathbf{h}-\mathbf{c})+(\mathbf{b}-\mathbf{c}).(\mathbf{h}-\mathbf{a})+(\mathbf{c}-\mathbf{a}).(\mathbf{h}-\mathbf{b})=0$$

show that the perpendiculars from the vertices A, B, C of a triangle ABC meet at a point H. (*Orthocentre of the triangle*).

(ii) From the identity

$$(\mathbf{a}-\mathbf{b}).\left(\mathbf{k}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)+(\mathbf{b}-\mathbf{c}).\left(\mathbf{k}-\frac{\mathbf{b}+\mathbf{c}}{2}\right)+(\mathbf{c}-\mathbf{a}).\left(\mathbf{k}-\frac{\mathbf{c}+\mathbf{a}}{2}\right)=0$$

show that the perpendicular bisectors of the triangle ABC meet in a point K (*Circum-centre of the triangle*).

(iii) If $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are equal in length, prove that

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OH}$$

where H is the orthocentre of the $\triangle ABC$.

22. Euler's Theorem. *In any triangle circum-centre, ortho-centre and centroid are collinear and the centroid divides the join of the other two in the ratio* 2 : 1.

Proof. Let K, G, H be the circum-centre, centroid and ortho-centre of the $\triangle ABC$ (Fig. 3.3). With reference to some

origin let the position vectors of A, B, C, K, G, H be **a**, **b**, **c**, **k**, **g**, **h** respectively. Since H is the orthocentre, HC is perpendicular to AB.

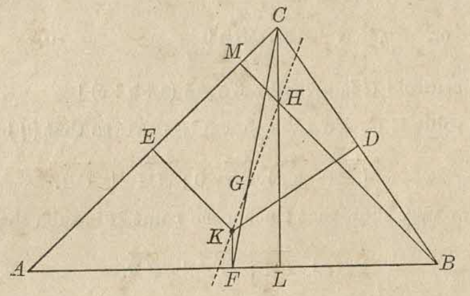


Fig. 3.3. Euler's Theorem

$$i.e., \quad (\mathbf{c}-\mathbf{h}).(\mathbf{b}-\mathbf{a})=0 \qquad \cdots \qquad (1)$$

Again since K is the circum-centre, KF is perpendicular to AB.

$$i.e., \quad \{\tfrac{1}{2}(\mathbf{a}+\mathbf{b})-\mathbf{k}\}.(\mathbf{b}-\mathbf{a})=0$$

$$\text{or,} \quad (\mathbf{a}+\mathbf{b}-2\mathbf{k}).(\mathbf{b}-\mathbf{a})=0 \qquad \cdots \qquad (2)$$

On adding (1) and (2), we obtain

$$(\mathbf{a}+\mathbf{b}+\mathbf{c}-2\mathbf{k}-\mathbf{h}).(\mathbf{b}-\mathbf{a})=0 \qquad \cdots \qquad (3)$$

This equation (3) holds when $\mathbf{b}-\mathbf{a}$ is replaced by $\mathbf{c}-\mathbf{b}$ and $\mathbf{a}-\mathbf{c}$, we conclude therefore,

$$\mathbf{a}+\mathbf{b}+\mathbf{c}-2\mathbf{k}-\mathbf{h}=\mathbf{0}.$$

For, otherwise $\mathbf{a}+\mathbf{b}+\mathbf{c}-2\mathbf{k}-\mathbf{h}$ would be perpendicular to three sides of $\triangle ABC$ which is impossible. The last relation shows

$$\mathbf{g}=\tfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})=\tfrac{1}{3}(\mathbf{h}+2\mathbf{k})$$

which proves the theorem.

Otherwise. Let the circum-centre K be taken as origin. Suppose with respect to this origin the position vectors of A, B, C be **a**, **b**, **c** respectively.

Since $KA=KB=KC$ we have $\mathbf{a}^2=\mathbf{b}^2=\mathbf{c}^2$. Now let us rewrite $\mathbf{b}^2-\mathbf{c}^2=0$ in the form

$$(\mathbf{b}+\mathbf{c}).(\mathbf{b}-\mathbf{c})=0$$

or, $$\{\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{a}\}.(\mathbf{b}-\mathbf{c})=0$$

or, $$\{3\overrightarrow{KG}-\overrightarrow{KA}\}.\overrightarrow{CB}=0 \qquad \cdots \qquad \text{(i)}$$

[$\because$ the centroid G is given by $\overrightarrow{KG}=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$]

We take a point H on KG such that $KH=3KG$ so that (i) becomes

$$(\overrightarrow{KH}-\overrightarrow{KA}).\overrightarrow{CB}=0\ ;\ \overrightarrow{AH}\perp\overrightarrow{CB}.$$

Similarly we may show that the chosen point H is such that

$$\overrightarrow{BH}\perp\overrightarrow{CA}\ ;\ \overrightarrow{CH}\perp\overrightarrow{AB}.$$

Thus H is the ortho-centre and it thus lies on the line KG and G divides KH in the ratio 2 : 1.

23. Lagrange's Theorem. *Particles of masses $m_1, m_2 \ldots m_r$ are placed at the points A, B, $C, \ldots K$ respectively and G is the centre of mass. Prove that for any point P,*

$$m_1AP^2+m_2BP^2+\cdots+m_rKP^2$$
$$=(m_1AG^2+m_2BG^2+\cdots+m_rKG^2)+(m_1+m_2+\cdots+m_r)PG^2.$$

Proof. Take G as origin and let $\mathbf{p}$ be the position vector of P. Suppose the position vectors of A, B, C, ... K be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... $\mathbf{k}$ respectively. Now, from definition of centre of mass and our choice of G it follows

$$m_1\mathbf{a}+m_2\mathbf{b}+\cdots+m_r\mathbf{k}=\mathbf{0} \qquad \cdots \qquad (1)$$

Since $\overrightarrow{AP}=\overrightarrow{GP}-\overrightarrow{GA}=\mathbf{p}-\mathbf{a}$, we have

$$m_1AP^2+m_2BP^2+\cdots+m_rKP^2$$
$$=m_1(\mathbf{p}-\mathbf{a})^2+m_2(\mathbf{p}-\mathbf{b})^2+\cdots+m_r(\mathbf{p}-\mathbf{k})^2$$
$$=(m_1+m_2+\cdots+m_r)\mathbf{p}^2+(m_1\mathbf{a}^2+m_2\mathbf{b}^2+\cdots+m_r\mathbf{k}^2)$$
$$-2\mathbf{p}.(m_1\mathbf{a}+m_2\mathbf{b}+\cdots+m_r\mathbf{k})$$

$=$ required result by virtue of (1).

Hints and Answers

1. By triangle law, $-\mathbf{c}=\mathbf{a}+\mathbf{b}$, (draw a diagram), and hence

$$(-\mathbf{c}).(-\mathbf{c})=(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a}^2+2\mathbf{a}.\mathbf{b}+\mathbf{b}^2, \text{ (distributive law).}$$

Since the square of a vector is the square of its length, it follows,

$$c^2=a^2+b^2+2ab\cos(\pi-C) \quad (\because \widehat{(\mathbf{a}, \mathbf{b})}=\pi-C).$$

2. Use (xiii) of art. 3·21. Here $\mathbf{a}.\mathbf{b}=10$, $|\mathbf{a}|^2=14$, $|\mathbf{b}|^2=20$, and hence $\theta=\cos^{-1} 10/\sqrt{280}=53°18'$ (approx.)

3. (*a*) $\cos^{-1}(12/77)$. (*b*) 1. $\cos^{-1} 1/3$. **4.** 14.

5. Σb_1c_1, Σc_1a_1, Σa_1b_1. **6.** Verify that $\alpha.\beta=\beta.\gamma=\gamma.\alpha=0$.

8. To find all the angles and lengths of all the sides of the triangle ABC.

$$\overrightarrow{AB}=(2, 1, -1)-(1, -2, 3)=(1, 3, -4) \text{ ; length}= \sqrt{26}$$

$$\overrightarrow{BC}=(3, -1, 2)-(2, 1, -1)=(1, -2, 3) \text{ ; length}= \sqrt{14}.$$

Now use $\overrightarrow{AB}.\overrightarrow{BC}=|\overrightarrow{AB}|\,|\overrightarrow{BC}|\cos(\pi-B)$ and hence obtain B.

Similarly obtain other angles and the remaining side.

9. If $\mathbf{l} \neq \mathbf{0}$ then $\mathbf{l}.\mathbf{a}=\mathbf{l}.\mathbf{b}=0$ implies that $\mathbf{l}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ *i.e.*, $\mathbf{l}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. But $\mathbf{l}.\mathbf{c}=0$ means $\mathbf{l}$ is perpendicular to $\mathbf{c}$. $\therefore$ $\mathbf{l}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ as well as to $\mathbf{c}$. This is equivalent to say that $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$. But $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar ; hence a contradiction.

10. With reference to an origin O, let the position vectors of the vertices A, B, C of the $\triangle ABC$ be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Let $\mathbf{p}$ be the position vector of P, the point of intersection of the *two* perpendiculars from A and B on the opposite sides. Since $\overrightarrow{AP} \perp \overrightarrow{BC}$ and $\overrightarrow{BP} \perp \overrightarrow{CA}$, we have

$$(\mathbf{p}-\mathbf{a}).(\mathbf{c}-\mathbf{b})=0 \text{ ; } (\mathbf{p}-\mathbf{b}).(\mathbf{a}-\mathbf{c})=0.$$

Expanding by distributive law and then adding, we get

$$(\mathbf{p}-\mathbf{c}).(\mathbf{a}-\mathbf{b})=0, \textit{ i.e., } \overrightarrow{CP} \perp \overrightarrow{BA}.$$

i.e., perpendicular from C on the opposite side goes through P also ; hence etc.

11. $\Sigma(c_1+d_1)^2=\Sigma(c_1-d_1)^2$, by the given condition. Expand.

12. Let H be the point of intersection of the perpendicular bisectors of the sides BC and CA. With usual notations for position vectors, we have then,

$$[\mathbf{h}-\tfrac{1}{2}(\mathbf{b}+\mathbf{c})].(\mathbf{b}-\mathbf{c})=0\ ;\ [\mathbf{h}-\tfrac{1}{2}(\mathbf{a}+\mathbf{c})].(\mathbf{a}-\mathbf{c})=0.$$

Expand by distributive law and then add. Thus obtain

$$[\mathbf{h}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})].(\mathbf{a}-\mathbf{b})=0$$

i.e., H lies on the perpendicular bisector of AB. Hence the result.

13. Comp. $\mathbf{a}$ (along $\mathbf{b}$) $=\hat{\mathbf{b}}.\mathbf{a}=|\mathbf{a}|\cos\theta$; obtain $\cos\theta$ from $\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

15. $\overrightarrow{BC}=(1, 1, -2)$, $\overrightarrow{AB}=(0, 1, 2)$; $|\overrightarrow{BC}|=\sqrt{6}$, $|\overrightarrow{AB}|=\sqrt{5}$;

Obtain $\cos\theta=-3/\sqrt{30}$ and hence $\sin\theta=\sqrt{7}/\sqrt{10}$,

$\therefore$ area $=\tfrac{1}{2}\,|\overrightarrow{BC}|\,|\overrightarrow{AB}|\sin\theta=\tfrac{1}{2}\sqrt{21}$.

16. $\mathbf{e}_1.\mathbf{e}_2=\cos\theta$ and write $2\mathbf{e}_1.\mathbf{e}_2=2-4\sin^2\theta/2$.

$\therefore\ 4\sin^2\theta/2=|\mathbf{e}_1|^2+|\mathbf{e}_2|^2-2\mathbf{e}_1.\mathbf{e}_2=|\mathbf{e}_1-\mathbf{e}_2|^2$; hence etc.

17. Refer to Fig. 3.4. Suppose $\overrightarrow{AB}\perp\overrightarrow{CD}$, $\overrightarrow{AD}\perp\overrightarrow{BC}$. With usual notations then

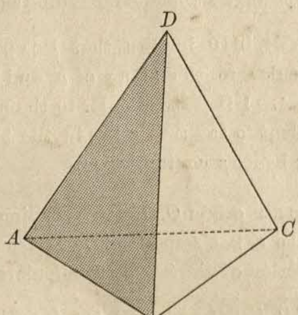


Fig. 3.4. Tetrahedron

$$(\mathbf{b}-\mathbf{a}).(\mathbf{d}-\mathbf{c})=0,\ (\mathbf{d}-\mathbf{a}).(\mathbf{c}-\mathbf{b})=0.$$

Add the results and obtain $(\mathbf{d}-\mathbf{b}).(\mathbf{c}-\mathbf{a})=0$, *i.e.*, $\overrightarrow{BD}\perp\overrightarrow{CA}$.

19. (i) Take two unit vectors $\mathbf{r}_1, \mathbf{r}_2$ making angles B, A with the x-axis. Then $\mathbf{r}_1=(\cos B, \sin B)$; $\mathbf{r}_2=(\cos A, \sin A)$.

Also $\mathbf{r}_1 \cdot \mathbf{r}_2 = \cos A \cos B + \sin A \sin B$, but $\mathbf{r}_1 . \mathbf{r}_2 = \cos (A-B)$.

(iii) $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\mathbf{0}$. Take scalar product with $\overrightarrow{AB}$.

Then $\overrightarrow{BC}.\overrightarrow{AB}+\overrightarrow{CA}.\overrightarrow{AB}=-\overrightarrow{AB}.\overrightarrow{AB}$

or, $ac \cos (\pi-B)+bc \cos (\pi-A)=-c^2$; hence etc.

20. Let P be any point on the semi-circle with centre at O and AOB as diameter. Then $\overrightarrow{AP}.\overrightarrow{BP}=(\mathbf{p}-\mathbf{a}).(\mathbf{p}-\mathbf{b})=\mathbf{p}^2-\mathbf{a}^2=0$; hence $AP \perp BP$.

21. (i), (ii) If two terms of the identity are zero, the third is so.

(iii) Multiply scalarly $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{h}$ by $\mathbf{a}-\mathbf{b}$, $\mathbf{b}-\mathbf{c}$, $\mathbf{c}-\mathbf{a}$ in turn.

3·3. Vector Product : Definition from Geometric Stand-point.

DEFINITION. The vector product of two vectors **a** and **b**, written as $\mathbf{a} \times \mathbf{b}$, is a vector **v** whose

(i) *length* is $|\mathbf{a}||\mathbf{b}| \sin \theta$, where θ is the smallest angle between **a** and **b** $(0 \leqslant \theta \leqslant \pi)$;

(ii) *support* is perpendicular to both **a** and **b** (*i.e.*, perpendicular to the plane containing **a** and **b**) and

(iii) *sense* is such that as we turn the first-named vector **a** towards the second vector **b** through an angle θ, **v** will point in the direction in which a right-handed screw would advance if turned in similar manner ; see Fig. 3.5.

In short, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \, \varepsilon$, where ε is a unit vector normal to the plane of **a** and **b** and the sense of ε is such that **a**, **b**, ε form a right-handed triad of vectors. Our definition implies that ε is not defined when $\theta = 0$. We agree to write $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if the angle between **a** and **b** be zero, (*i.e.*, if **a** and **b** are parallel).

Note. $\mathbf{a} \times \mathbf{b}$ is a *vector*. It will be also called *cross product* or *outer product* of **a** and **b**. Some prefer to use the notation $\mathbf{a} \wedge \mathbf{b}$ for the vector product of **a** and **b**.

Motivation. We have observed before that a force is an example of a vector quantity localised along a straight line. The directed segment **F** used to represent the force, gives only its magnitude and direction but does not give its position in space. To specify the definite position we require another vector besides **F**. Let O be any convenient point, and **r** be the position vector

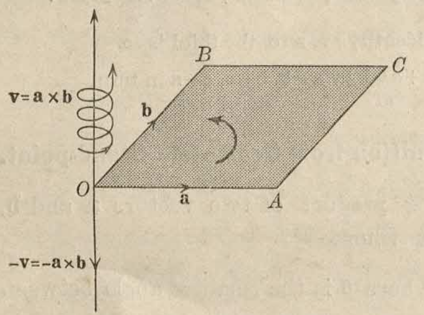


Fig. 3.5. Vector product

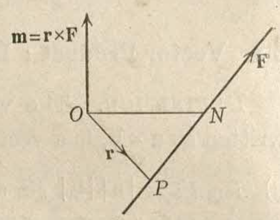


Fig. 3.6. Moment of a force

relative to O of any point P on the line of the force (Fig. 3.6). Then *moment of the force about the point* O is defined by the vector $\mathbf{m}=\mathbf{r}\times\mathbf{F}$. The vector **m** is thus perpendicular to the plane of **r** and **F** and hence perpendicular to the plane containing O and the line of **F**. Its magnitude is $p|\mathbf{F}|$, where p is the length of the perpendicular ON from O to the line of action of **F**. Conversely, given **F** and **m**, the force is specified in magnitude, direction and position. The line of action of **F** thus lies in the plane through O perpendicular to **m**, which is the plane OPN. Its direction is that of **F** and its distance p from O be such that $p|\mathbf{F}|=|\mathbf{m}|$.

It lies on that side of O which makes a rotation from OP to **F** *positive* (anticlockwise) as seen from the terminal point of **m**.

3·31. Properties of Vector Products.

From the definition given above we may deduce the following properties :

(i) **Vector product is non-commutative (in fact, anti-commutative) :**

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a} \text{ (rather, } \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a})$$

For, $\mathbf{b} \times \mathbf{a}$ is a vector $\mathbf{v}'$ whose length is $|\mathbf{b}|\,|\mathbf{a}| \sin \theta$, support is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{a}$ but whose sense is such that *as we rotate* $\mathbf{b}$ *towards* $\mathbf{a}$, $\mathbf{v}'$ *will point in the direction that a right-handed screw will advance if turned in a similar manner* (Fig. 3.5). Clearly sense of $\mathbf{v}'$ will be opposite to that of $\mathbf{b} \times \mathbf{a}$, though their lengths (*viz.* $|\mathbf{b}|\,|\mathbf{a}| \sin \theta$) are same. Hence $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(ii) We have defined

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin \theta\ \varepsilon, \quad \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ are not parallel,}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}, \quad \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel.}$$

Hence $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ may lead to one of the four following conclusions :

(a) $\mathbf{a}$ is a null vector ; (b) $\mathbf{b}$ is a null vector ;

(c) $\mathbf{a}$ and $\mathbf{b}$ are both null vectors ; (d) $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Conclusion : *For two proper vectors* $\mathbf{a}$ *and* $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ *implies that* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.* This is the condition of parallelism of two non-zero vectors.

In particular,

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}\ ;\ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

CAUTION. We should never write $\mathbf{a} \times \mathbf{a} = \mathbf{a}^2$, since we have already agreed to mean $\mathbf{a}^2 = \mathbf{a}.\mathbf{a}$.

(iii) If $\mathbf{a}$ and $\mathbf{b}$ are at right angles, then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\varepsilon$ where ε is a unit vector in the direction perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ and its sense is such that

$$\mathbf{a},\ \mathbf{b},\ \mathbf{a} \times \mathbf{b}$$

form a right-handed triad of three mutually perpendicular vectors. As a familiar illustration, we have for the three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the coördinate directions forming a right-handed system of axes the following relations :

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}\ ;\ \mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j}\ ;\ \mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}.$$

(iv) Associativity with Scalars :

$$(m\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(m\mathbf{b})=m\,(\mathbf{a}\times\mathbf{b}).$$

i.e., if either factor be multiplied by a number m then the vector is multiplied by that number.

For, $(m\mathbf{a})\times\mathbf{b}=|m\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\varepsilon$

$$=|\mathbf{a}|\{|m\mathbf{b}|\sin\theta\}\,\varepsilon=\mathbf{a}\times(m\mathbf{b})$$

$$=|m|\{|\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\varepsilon\}=m(\mathbf{a}\times\mathbf{b}).$$

More generally, $(m\mathbf{a})\times(n\mathbf{b})=mn\,(\mathbf{a}\times\mathbf{b})$, where m and n are any two numbers, positive or negative.

Associativity with Vectors :

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are both defined but they are not, in general, equal.

(v) An important relation connecting $\mathbf{a}\times\mathbf{b}$ with $\mathbf{a}.\mathbf{b}$:

$$(\mathbf{a}\times\mathbf{b})^2=\mathbf{a}^2\,\mathbf{b}^2-(\mathbf{a}.\mathbf{b})^2.$$

For, $(\mathbf{a}\times\mathbf{b})^2=(\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})=|\mathbf{a}\times\mathbf{b}|^2$

$$=|\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta=|\mathbf{a}|^2\,|\mathbf{b}|^2\,(1-\cos^2\theta)$$

$$=|\mathbf{a}|^2\,|\mathbf{b}|^2-|\mathbf{a}|^2\,|\mathbf{b}|^2\cos^2\theta$$

$$=\mathbf{a}^2\,\mathbf{b}^2-(\mathbf{a}.\mathbf{b})^2.$$

Note that we have used the fact that the *square of a vector is equal to square of its length.*

(vi) Distributive Law :

For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} \qquad \cdots \quad (A)$$

i.e., vector product is distributive with respect to addition.

Secondly,

$$(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a} \qquad \cdots \quad \text{(B)}$$

(This second distributive law is distinct from the first since vector product is not commutative).

Proof. We shall give a geometrical construction for the vector product $\mathbf{a}\times\mathbf{b}$ which will demonstrate the truth of the distributive law directly.

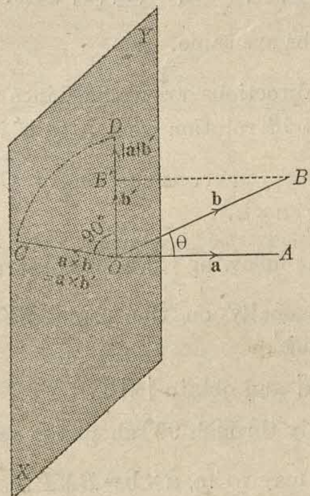


Fig. 3.7. Vector product $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}'$

First Step. Let the two vectors $\overrightarrow{OA}\,(=\mathbf{a})$ and $\overrightarrow{OB}\,(=\mathbf{b})$ be inclined at an angle θ (Fig. 3.7). Through O draw a plane XY perpendicular to $\mathbf{a}$.

We project $\mathbf{b}$ orthogonally on this plane, thus obtaining the vector $\overrightarrow{OB'}=\mathbf{b}'$. Clearly $|\mathbf{b}'|=|\mathbf{b}|\sin\theta$. Next consider the vector $\overrightarrow{OD}=|\mathbf{a}|\mathbf{b}'$ *i.e.*, the vector $\mathbf{b}'$ multiplied by the number $|\mathbf{a}|$.

Now rotate this vector $|\mathbf{a}|\mathbf{b}'$ about OA in the positive sense through a right angle. Suppose the resulting vector is $\overrightarrow{OC}$. We now verify :

$$\mathbf{a}\times\mathbf{b}'=\mathbf{a}\times\mathbf{b}=\overrightarrow{OC}.$$

Evidently,

$$|\mathbf{a}\times\mathbf{b}'|=|\mathbf{a}|\,|\mathbf{b}'|\sin 90^\circ=|\mathbf{a}|\,|\mathbf{b}|\sin\theta\ (\because |\mathbf{b}'|=|\mathbf{b}|\sin\theta)$$

Also, by definition, $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\sin\theta$.

Thus their lengths are same.

Further, their directions are same, since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{b}'$ lie in *one plane* and the sense of rotation from $\mathbf{a}$ to $\mathbf{b}'$ is the same as that from $\mathbf{a}$ to $\mathbf{b}$. Hence our construction of $\overrightarrow{OC}$ shows that it represents $\mathbf{a}\times\mathbf{b}'$ or $\mathbf{a}\times\mathbf{b}$.

Thus we have the following rules for constructing $\mathbf{a}\times\mathbf{b}$:

1. Project $\mathbf{b}$ orthogonally on the plane XY perpendicular to $\mathbf{a}$ at O and obtain $\mathbf{b}'$.
2. Multiply $\mathbf{b}'$ by $|\mathbf{a}|$ and obtain $|\mathbf{a}|\mathbf{b}'$.
3. Rotate it positively through 90° about the vector $\mathbf{a}$.

In notations, we may write, $\mathbf{a}\times\mathbf{b}=RMP\,\mathbf{b}$ ... (C)

This means that $\mathbf{b}$ is *projected* (P), projection *multiplied* (M) and finally *rotated* (R).

Second Step. Each of the operations R, M, P is distributive *i.e.*, operating on the sum of two vectors is the same as operating on the vectors separately and adding the results ; hence

$$RMP(\mathbf{b}+\mathbf{c})=RM(P\mathbf{b}+P\mathbf{c})=R(MP\mathbf{b}+MP\mathbf{c})$$
$$=RMP\mathbf{b}+RMP\mathbf{c}.$$

Hence using (C), $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.

Otherwise. We may argue as :

Suppose **b** and **c** are represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ (Fig. 3.8) of a parallelogram $OBDC$, whose diagonal OD is the sum $\mathbf{b}+\mathbf{c}$. We now perform the operations P, M, R on the parallelogram $OBDC$ instead of on the individual vectors $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}+\mathbf{c}$.

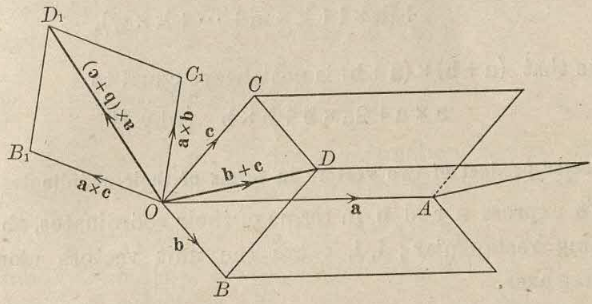


Fig. 3.8. Distributive law of vector product

We thus obtain a parallelogram $OB_1D_1C_1$ whose sides $\overrightarrow{OB_1}$, $\overrightarrow{OC_1}$ represent vectors $\mathbf{a}\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b}$ and whose diagonal is the product $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$. From this, the relation

$$\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}+\mathbf{c})$$

clearly follows.

Proof of second distributive law :

$$(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}. \qquad \ldots \quad (\mathbf{B})$$

This follows from the first distributive law (**A**) when commutative rule (**i**) is applied.

Repeated application of this result shows that we may expand the vector product of two sums as in ordinary algebra, *provided the order of the factors is not altered.* Thus

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d})&=(\mathbf{a}+\mathbf{b})\times\mathbf{c}+(\mathbf{a}+\mathbf{b})\times\mathbf{d}\\&=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{d}+\mathbf{b}\times\mathbf{d}.\end{aligned}$$

More generally,

$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\cdots+\mathbf{k})\times(\mathbf{l}+\mathbf{m}+\cdots+\mathbf{w}) &= (\mathbf{a}\times\mathbf{l}+\mathbf{a}\times\mathbf{m}+\cdots+\mathbf{a}\times\mathbf{w})\\ &\quad+(\mathbf{b}\times\mathbf{l}+\mathbf{b}\times\mathbf{m}+\cdots+\mathbf{b}\times\mathbf{w})\\ &\quad+\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad+(\mathbf{k}\times\mathbf{l}+\mathbf{k}\times\mathbf{m}+\cdots+\mathbf{k}\times\mathbf{w}).\end{aligned}$$

Note that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b})$ is not always equal to

$$\mathbf{a}\times\mathbf{a}+2\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b} \quad (\text{why ?})$$

(vii) Vector product of two vectors in terms of their coördinates :

We express $\mathbf{a}$ and $\mathbf{b}$ in terms of their coördinates, the axes being rectangular ; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the three axes.

Thus $\quad \mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}.$

Hence,

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i}\times b_1\mathbf{i})+(a_1\mathbf{i}\times b_2\mathbf{j})+(a_1\mathbf{i}\times b_3\mathbf{k})\\ &\quad+(a_2\mathbf{j}\times b_1\mathbf{i})+(a_2\mathbf{j}\times b_2\mathbf{j})+(a_2\mathbf{j}\times b_3\mathbf{k})\\ &\quad+(a_3\mathbf{k}\times b_1\mathbf{i})+(a_3\mathbf{k}\times b_2\mathbf{j})+(a_3\mathbf{k}\times b_3\mathbf{k}).\end{aligned}$$

(Distributive Law)

$$\begin{aligned}&=(a_1b_2-a_2b_1)\,\mathbf{i}\times\mathbf{j}+(a_3b_1-a_1b_3)\,\mathbf{k}\times\mathbf{i}\\ &\qquad\qquad+(a_2b_3-a_3b_2)\,\mathbf{j}\times\mathbf{k}\\ &=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k},\end{aligned}$$

[using (ii) and (iii) above]

The coördinates of the vector $\mathbf{a}\times\mathbf{b}$ are, therefore,

$$a_2b_3-a_3b_2=\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}, \qquad a_3b_1-a_1b_3=\begin{vmatrix} a_3 & a_1\\ b_3 & b_1\end{vmatrix},$$

$$a_1b_2-a_2b_1=\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}.$$

and consequently we write in a symmetrical form :

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note that the square of $\mathbf{a} \times \mathbf{b} = |\mathbf{a} \times \mathbf{b}|^2 = \Sigma(a_2 b_3 - a_3 b_2)^2$, whence, it follows (since $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \theta$),

$$\sin^2\theta = \frac{\Sigma(a_2 b_3 - a_3 b_2)^2}{\Sigma a_1^{\,2}\ \Sigma b_1^{\,2}}.$$

If $(l, m, n), (l', m', n')$ be the direction cosines of $\mathbf{a}$ and $\mathbf{b}$ respectively then this gives $\sin^2\theta = \Sigma\,(mn' - m'n)^2$.

For, $l = a_1/|\mathbf{a}|$, $m = a_2/|\mathbf{a}|$, $n = a_3/|\mathbf{a}|$;

and $l' = b_1/|\mathbf{b}|$, $m' = b_2/|\mathbf{b}|$, $n' = b_3/|\mathbf{b}|$.

Note. The students should make a practice of obtaining the vector product of two vectors whose components are given. Thus to find $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, one should form the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & 7 \\ 4 & 2 & 1 \end{vmatrix}$$

and expand in terms of first row. Thus $\mathbf{a} \times \mathbf{b} = -9\mathbf{i} + 25\mathbf{j} - 14\mathbf{k}$.

(viii) If $\mathbf{b} = \mathbf{c} + n\mathbf{a}$, where n is a scalar, then

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{c} + n\mathbf{a})$$
$$= \mathbf{a} \times \mathbf{c} + \mathbf{a} \times (n\mathbf{a}) = \mathbf{a} \times \mathbf{c} \quad [\because\ \mathbf{a} \times (n\mathbf{a}) = \mathbf{0}\,]$$

Conversely, If $\mathbf{a} \neq \mathbf{0}$ then from $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ [or $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$] we should not at once conclude that $\mathbf{b} = \mathbf{c}$; it may happen that $\mathbf{b} - \mathbf{c}$ and $\mathbf{a}$ are parallel (*i.e.*, $\mathbf{b}$ may differ from $\mathbf{c}$ by a vector parallel to $\mathbf{a}$, say $n\mathbf{a}$).

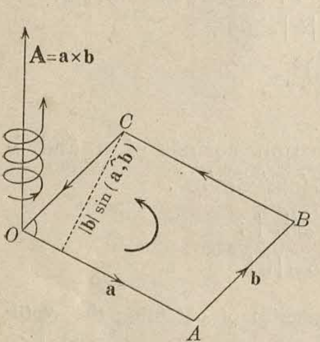
(ix) Vector area : Vector product as vector area.

DEFINITION. A plane area bounded by a closed figure (without multiple points) and traced in a definite sense corresponds to a vector $\mathbf{A}$, called *vector area*, defined as follows :

(*a*) length of **A** is equal to the *number of units of area* of the given figure.

(*b*) support of **A** is perpendicular to the plane of the area.

(*c*) sense of **A** is such that the direction of description of the boundary and sense of **A** corresponds to the rotation and translation respectively of a right-handed screw.



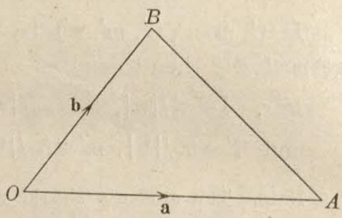


Fig. 3.9(*a*) Vector area of a parallelogram Fig. 3.9(*b*). Vector area of a triangle

We may now interpret $\mathbf{a} \times \mathbf{b}$ as a vector area **A** of the parallelogram whose sides represent **a** and **b**. For,

Area A of the parallelogram $= |\mathbf{a}|\,|\mathbf{b}| \sin(\widehat{\mathbf{a}, \mathbf{b}})$.

Sense of **A** and $\mathbf{a} \times \mathbf{b}$ are same (as will be clear from Fig. 3.9*a*).

We may also interpret

$$2\,[\text{vector area of } \triangle OAB] = \mathbf{a} \times \mathbf{b}\,;$$

where $\mathbf{a} = \overrightarrow{OA}$, and $\mathbf{b} = \overrightarrow{OB}$ (Fig. 3.9*b*)

3·32. Vector product from Algebraic stand-point.

DEFINITION. If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors given in terms of number triples, then their cross-product $\mathbf{a} \times \mathbf{b}$ is defined by the vector

$$(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

Rule. *The number triples of* $\mathbf{a} \times \mathbf{b}$ *are determinants formed by columns* 2 *and* 3, 3 *and* 1 *(Not* 1 *and* 3*)*, 1 *and* 2 *of the array*

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Geometric interpretation :

1. When $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, it represents a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; For,

$$\begin{aligned} \mathbf{a}.(\mathbf{a} \times \mathbf{b}) &= (a_1, a_2, a_3).(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\ &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= 0. \end{aligned}$$

and similarly, $\mathbf{b}.(\mathbf{a} \times \mathbf{b}) = 0$. Hence $\mathbf{a} \times \mathbf{b} \perp$ to both $\mathbf{a}$ and $\mathbf{b}$.

2. $|\mathbf{a} \times \mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$;

$|\mathbf{a}|^2 = a_1^{\,2} + a_2^{\,2} + a_3^{\,2}$ and $|\mathbf{b}|^2 = b_1^{\,2} + b_2^{\,2} + b_3^{\,2}$;

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

With these results we can establish the identity

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2.$$

Hence $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta.$$

$$\therefore \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \; |\mathbf{b}| \sin\theta.$$

This gives the magnitude of $\mathbf{a} \times \mathbf{b}$.

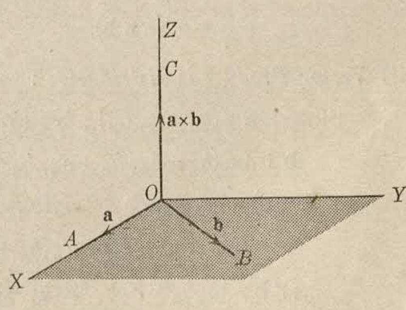


Fig. 3.10 To fix the direction of $\mathbf{a} \times \mathbf{b}$

3. To find the direction of $\mathbf{a} \times \mathbf{b}$, shift $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ to the origin and revolve the trihedral $O-xyz$ so that x-axis points along $\mathbf{a}$ and the y-axis lies in the plane of $\mathbf{a}$ and $\mathbf{b}$ and makes an acute angle with $\mathbf{b}$ (Fig. 3.10). In course of this continuous

rotation $\mathbf{a} \times \mathbf{b}$ remains *fixed.* For, it is a vector of known length perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ and continuity forbids a change in direction. Now,

$$\mathbf{a} = (|\mathbf{a}|, 0, 0)\ ;\ \mathbf{b} = (|\mathbf{b}|\cos\theta,\ |\mathbf{b}|\sin\theta,\ 0)$$

$$\mathbf{a} \times \mathbf{b} = (0,\ 0,\ |\mathbf{a}|\,|\mathbf{b}|\sin\theta).$$

Hence $\mathbf{a} \times \mathbf{b}$ points along the positive z-axis and since the axes are right-handed, the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ also form a right-handed vector triad.

Conclusion : The product $\mathbf{a} \times \mathbf{b}$ as defined before by number triples is a vector of length $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$ perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ and pointing in the direction that a right-handed screw will advance when turned from $\mathbf{a}$ towards $\mathbf{b}$. Thus this definition is equivalent to the definition of art. 3·3.

Properties of vector-products : All the properties discussed in *art.* 3·31 can be deduced from the definition

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1).$$

We show, in particular, the truth of the following two laws :

(i) *Vector Product is anti-commutative* :

$$\text{For,}\quad \mathbf{b} \times \mathbf{a} = (b_2a_3 - b_3a_2,\ b_3a_1 - b_1a_3,\ b_1a_2 - b_2a_1)$$
$$= -(a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$
$$= -(\mathbf{a} \times \mathbf{b}).$$

(ii) *Vector Product is distributive* : $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Since $\mathbf{a} = (a_1, a_2, a_3)$; $\mathbf{b} = (b_1, b_2, b_3)$; $\mathbf{c} = (c_1, c_2, c_3)$

$$\mathbf{b} + \mathbf{c} = (b_1 + c_1,\ b_2 + c_2,\ b_3 + c_3).$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \{a_2(b_3 + c_3) - a_3(b_2 + c_2),\ a_3(b_1 + c_1) - a_1(b_3 + c_3),$$
$$a_1(b_2 + c_2) - a_2(b_1 + c_1)\}$$

$$\mathbf{a} \times \mathbf{b} = \{a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\}$$

$$\mathbf{a} \times \mathbf{c} = \{a_2c_3 - a_3c_2,\ a_3c_1 - a_1c_3,\ a_1c_2 - a_2c_1\}$$

Now we can easily verify $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

Examples. III(B)

1. The vectors **a** and **b** are given by

$$\mathbf{a}=2\mathbf{i}+3\mathbf{j}+6\mathbf{k}\ ;\ \mathbf{b}=3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\ ;\ \text{obtain } \mathbf{a}\times\mathbf{b}.$$

2. Show that $(2, 5, -7)\times(-3, 4, 1)=(33, 19, 23)$.

3. The vectors from the origin to the points A, B, C are

$$\mathbf{a}=(2, -1, 1)\ ;\ \mathbf{b}=(3, 0, 1)\ ;\ \mathbf{c}=(1, -2, 3).$$

Find the unit vector **v** perpendicular to the plane ABC.

4. Find the length of the vector $(3\mathbf{i}+4\mathbf{j})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})$.

5. Show that the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})$ is coplanar with **a** and **b**.

6. Find the unit vector perpendicular to each of the vectors $\mathbf{a}(1, -3, 4)$ and $\mathbf{b}(2, -5, 3)$ and the sine of the angle between these vectors.

7. If **a**, **b**, **c** are vectors from the origin to the points A, B, C, show that $\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane ABC.

8. Prove that three points **a**, **b**, **c** are collinear if

$$\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}=\mathbf{0}.$$

9. Show that the laws of refraction of light passing from a medium, of refractive index μ into one of index μ' is expressed by $\mu\mathbf{n}\times\mathbf{u}=\mu'\mathbf{n}\times\mathbf{u}'$, where $\mathbf{n}$, $\mathbf{u}$, $\mathbf{u}'$ are unit vectors perpendicular to the boundary, along the incident ray and along the refracted ray respectively.

What does $\mathbf{u}\times\mathbf{n}=\mathbf{b}\times\mathbf{n}$ signify if **b** is the unit vector along the reflected ray ?

10. If two nonzero vectors $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\ ;\ \mathbf{r}'=x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}$$

(where **i**, **j**, **k** are a set of non-coplanar vectors and x, y, z ; x', y', z' are scalars), prove that the necessary and sufficient conditions of parallelism are $x : y : z=x' : y' : z'$.

Choose the scalars u and v so that the two vectors

$$(u-1)\mathbf{i}+9\mathbf{j}+3\mathbf{k} \text{ and } 2\mathbf{i}+(v+4)\mathbf{j}+\mathbf{k}$$

may be *co-directional*.

11. (*a*) Find the area of the triangle ABC, the position vectors of whose vertices are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(*b*) Find the area of the triangle OAB formed by the two points

$$A : \mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ ;\ B : \mathbf{b} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}\ ;$$

O being the origin.

12. Find the area of the parallelogram determined by OA and OB, where $\mathbf{a}$ and $\mathbf{b}$ have the same expressions as in the previous example.

13. Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$. Interpret geometrically.

14. The vectors from the origin to the points A, B, C, D are

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}\ ;\ \mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$$
$$\mathbf{c} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\ ;\ \mathbf{d} = \mathbf{k} - \mathbf{j}.$$

Show that the lines AB and CD are parallel and find the ratio of their lengths.

15. Given $\mathbf{a} = \frac{1}{7}(2, 3, 6)$; $\mathbf{b} = \frac{1}{7}(3, -6, 2)$; $\mathbf{c} = \frac{1}{7}(6, 2, -3)$; shew that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are each of unit length, mutually perpendicular and $\mathbf{a} \times \mathbf{b} = \mathbf{c}$.

16. Given $\hat{\mathbf{a}} = (1, 0, 0)$ and $\hat{\mathbf{b}} = (0, 1, 0)$, shew that

$$\sin \tfrac{1}{2}\theta = \tfrac{1}{2} | \hat{\mathbf{b}} - \hat{\mathbf{a}} |,$$

where θ is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

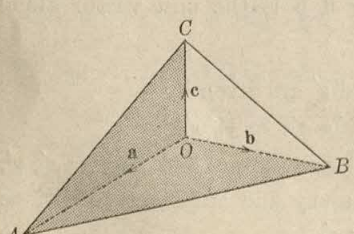


Fig. 3.11 Tetrahedron

17. Show that the sum of the vector areas, taken outwards, of the faces of the tetrahedron $OABC$ is zero. (Fig. 3.11).

18. In the triangle ABC, the position vectors of A, B, C are

a, **b**, **c** and the lengths of the sides BC, CA, AB are a, b, c respectively. Show that

(i) $\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$ is normal to the plane of ABC ;

(ii) $\frac{1}{2}bc \sin A=\frac{1}{2}|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|$;

(iii) $\sin A : a=\sin B : b=\sin C : c$

(A, B, C are the three angles of the $\triangle ABC$).

19. If A, B, C, D are any four points in space, show that

$$\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}$$

is independent of D.

20. ABC is a triangle. A' divides the side BC in the ratio $1:2$. B', C' are similar points on CA, AB. The pairs of lines

$$(AA', BB')\ ;\ (BB', CC')\ ;\ (CC', AA')$$

intersect at L, M, N respectively. Show that the area of $\triangle LMN=\frac{1}{7}$ area of $\triangle ABC$.

Hints and Answers

1. $42\mathbf{i}+14\mathbf{j}-21\mathbf{k}$.

3. $\overrightarrow{BC}=(-2, -2, 2)$, $\overrightarrow{BA}=(-1, -1, 0)$.

Hence $\overrightarrow{BC}\times\overrightarrow{BA}=(-2, -2, 2)\times(-1, -1, 0)=(2, -2, 0)$.

$\therefore$ Unit vector $\mathbf{v}$ perpendicular to the plane is $=\frac{1}{\sqrt{8}}(2, -2, 0)$.

4. $\sqrt{74}$.

5. The vector $\mathbf{b}\times\mathbf{a}$ is perpendicular to the plane containing **b** and **a**. Also $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})$ is perpendicular to both **a** and $\mathbf{b}\times\mathbf{a}$ and as such it will itself lie in the plane of **a** and **b**.

6. $\frac{1}{\sqrt{147}}(11, 5, 1)$; $\sin^2\theta=|\mathbf{a}\times\mathbf{b}|^2/|\mathbf{a}|^2\,|\mathbf{b}|^2=147/26.38$; hence etc.

7. $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$, $\overrightarrow{BA}=\mathbf{a}-\mathbf{b}$; hence we have

$$\begin{aligned}\overrightarrow{BC}\times\overrightarrow{BA}&=(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\mathbf{c}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}\\&=\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\quad(\because\ \mathbf{b}\times\mathbf{b}=\mathbf{0})\\&=\text{given vector.}\end{aligned}$$

Thus the given vector is perp. to BC as well as BA *i.e.*, perp. to the plane ABC.

9. *First part.* The equation expresses that the incident and refracted rays are coplanar with the normal to the surface of separation of the two media and also gives $\mu \sin i = \mu' \sin r$, where i and r are the angles of incidence and refraction respectively.

Second part. The incident and reflected rays are coplanar with the normal to the surface of separation and also gives that the angle of incidence and reflection are equal.

10. If $\mathbf{r}$ and $\mathbf{r}'$ are proper vectors, then the condition of parallelism gives $\mathbf{r} \times \mathbf{r}' = \mathbf{0}$. Hence for parallelism it necessarily follows

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) = 0,$$

$$i.e., \quad (yz' - zy')\, \mathbf{j} \times \mathbf{k} + (zx' - xz')\, \mathbf{k} \times \mathbf{i} + (xy' - x'y)\, \mathbf{i} \times \mathbf{j} = \mathbf{0} \quad \cdots \quad (1)$$

Since $\mathbf{j} \times \mathbf{k}$, $\mathbf{k} \times \mathbf{i}$, $\mathbf{i} \times \mathbf{j}$ are non-coplanar, three scalar coefficients must separately vanish so that

$$yz' - zy' = 0, \quad zx' - xz' = 0, \quad xy' - x'y = 0$$

which gives $\quad x : y : z = x' : y' : z'. \quad \cdots \quad \cdots \quad \cdots \quad (2)$

Conversely, if (2) holds, then suppose

$$x = \lambda x', \quad y = \lambda y' \quad z = \lambda z' \quad (\lambda = \text{a constant}).$$

These values of x, y, z will make the left side of (1) vanish so that $\mathbf{r} \times \mathbf{r}' = \mathbf{0}$ which will imply $\mathbf{r} \parallel \mathbf{r}'$.

Second part. Co-directional, if and only if $(u-1) : 2 = 9 : (v+4) = 3 : 1$ whence $u = 7, v = -3$.

11. (*a*) Area of the parallelogram given by AB and AC is the length of the vector $\quad \overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$.

$\therefore$ area of the triangle $ABC = \frac{1}{2}$ area of the above parallelogram

$$= \tfrac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \tfrac{1}{2} |\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}|$$

(*b*) Here area $= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = 3\sqrt{5}$.

12. $6\sqrt{5}$. **13.** Area of a parallelogram is half of the area of the parallelogram formed by the diagonals of the first parallelogram. This is the geometric interpretation required.

14. Verify $\overrightarrow{AB} \times \overrightarrow{CD} = \mathbf{0}$.

17. The vector area of the face ABC (Fig. 3.11) is found as in Ex. 11(a)

$$=\tfrac{1}{2}(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})$$

where the origin O is considered to be below the plane of ABC. The three terms of the last expression are the vector areas of the faces OBC, OCA, OAB measured *inwards*. Hence the result.

3·4. Product of three vectors.

From the three vectors **a, b, c** the following combinations may be derived :

1. $\mathbf{a}(\mathbf{b}.\mathbf{c})$ (a vector) ; multiplication of the vector **a** by a scalar $\mathbf{b}.\mathbf{c}$.
2. $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ (a scalar) ; dot-product of **a** and $\mathbf{b}\times\mathbf{c}$.
3. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ (a vector) ; vector-product of **a** and $\mathbf{b}\times\mathbf{c}$.
4. $\mathbf{a}\,(\mathbf{b}\times\mathbf{c})$ (not defined).
5. $\mathbf{a}.\,(\mathbf{b}.\mathbf{c})$ (absurd) ; dot-product requires two vectors and here we have a vector **a** and a scalar **b.c** on each side of the dot.
6. $\mathbf{a}\times(\mathbf{b}.\mathbf{c})$ (absurd) ; similar arguments as in 5.

We shall now consider only two of the above combinations *viz.* $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ (called *scalar-triple product* or *box-product*) and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ (called *Vector-triple product*). The box-product $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ is also denoted by the symbol

$$[\mathbf{abc}]\ ;$$

read as *box* **a, b, c**.

3·41. Scalar triple product.

Theorem. *The scalar triple product* $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ *or* $[\mathbf{abc}]$ *is a scalar and is numerically equal to the volume V of the parallelo-piped of which the three concurrent edges are* **a, b, c.** *Its sign is positive or negative according as* **a, b, c** *form a right-handed or a left-handed triad of vectors.*

Proof. We may write the vector product

$$\mathbf{b}\times\mathbf{c}=A\,\varepsilon,$$

where $A = |\mathbf{b} \times \mathbf{c}| = |\mathbf{b}|\,|\mathbf{c}| \sin \phi$ (ϕ being the smallest angle between **b** and **c**) and ε is a unit vector perpendicular to **b** and **c**.

Clearly A is the area of the parallelogram $OBDC$ having **b** and **c** as adjacent sides. Also ε is a unit vector perpendicular to the plane of the parallelogram and it points in a direction as a right-handed screw would advance when turned from **b** to **c**.

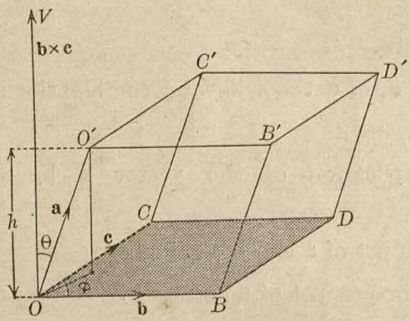


Fig. 3.12 Box product [**a b c**]

If the set **a**, **b**, **c** form a right-handed system (as in Fig. 3.12) the angle θ between ε and **a** is acute and then

$$\mathbf{a}.\mathbf{b} \times \mathbf{c} = \mathbf{a}.A\varepsilon = A|\mathbf{a}| \cos \theta = Ah = V$$

where h is the altitude of the parallelopiped.

When the set **a**, **b**, **c** form a left-handed system, θ is obtuse, $\cos \theta$ is negative and

$$\mathbf{a}.\mathbf{b} \times \mathbf{c} = -V$$

Note. The above theorem gives a geometric interpretation of the scalar triple product $\mathbf{a}.\mathbf{b} \times \mathbf{c}$; the name *box product* will now be significant.

Other Scalar triple products.

The volume of the parallelopiped may also be obtained by forming the vector products of any two of the three vectors **a, b, c** and then taking the dot-product with the remaining third vector. The triple product will be positive or negative according as **a, b, c** form a right-handed or a left-handed system of vectors. If, however, the order is changed, the sign of the product is also changed. For

$$\mathbf{b} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b}\ ;\ \mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}\ ;\ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

It follows that the volume of the parallelopiped

$$\left.\begin{aligned} &= \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = (\mathbf{c}\times\mathbf{a}).\mathbf{b} = -\mathbf{b}.(\mathbf{a}\times\mathbf{c}) = -(\mathbf{a}\times\mathbf{c}).\mathbf{b} \\ &= \mathbf{c}.(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b}).\mathbf{c} = -\mathbf{c}.(\mathbf{b}\times\mathbf{a}) = -(\mathbf{b}\times\mathbf{a}).\mathbf{c} \\ &= \mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c}).\mathbf{a} = -\mathbf{a}.(\mathbf{c}\times\mathbf{b}) = -(\mathbf{c}\times\mathbf{b}).\mathbf{a} \end{aligned}\right\} \quad (1)$$

Thus we have the following laws :

1. *The sign of the scalar triple product is unchanged as long as the cyclical order of the factors remains unchanged.*
2. *For every change of cyclical order a minus sign is introduced.*
3. *The dot and cross may be interchanged at pleasure i.e., the product is independent of the position of dot and cross.*

The equalities (1) that we have established above are called *Parallelopiped Law.*

Note. In box-notations, (1) may be written as

$$[\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{bac}] = -[\mathbf{acb}] = \text{etc.}$$

Since $(\mathbf{a}.\mathbf{b})\times\mathbf{c}$ or $\mathbf{a}\times(\mathbf{b}.\mathbf{c})$ are meaningless the paranthesis in $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ or $(\mathbf{a}\times\mathbf{b}).\mathbf{c}$ is unnecessary.

Scalar triple product when the vectors are given as number-triples.

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$.

Then $\mathbf{b}\times\mathbf{c} = (b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1)$.

Hence $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3)$
$+ a_3(b_1c_2 - b_2c_1)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This is the familiar expression for the volume of a parallelopiped with one corner at the origin and other corners with rectangular coördinates (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3).

In case the vectors are given in terms of rectangular components, say,

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\ ;\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\ ;\ \mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$$

we would obtain the same determinant for $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$.

More generally, if the three vectors are expressed in terms of three non-coplanar vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, we write

$$\mathbf{a}=a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n}\ ;\ \mathbf{b}=b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n}\ ;\ \mathbf{c}=c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n}$$

it can be easily shown that

$$[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\mathbf{l\ m\ n}]$$

Note. From the theory of determinants we know

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=-\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

They imply $[\mathbf{bca}]=[\mathbf{cab}]=-[\mathbf{bac}]=-[\mathbf{acb}]$.

Other relations of the *Parallelopiped Law* can be similarly deduced.

3·42. Condition of Coplanarity of three vectors.

Theorem. *If the three non-zero vectors are coplanar, their scalar triple product is zero.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-zero coplanar vectors. $\mathbf{b}\times\mathbf{c}$ is a vector perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. Since $\mathbf{a}$ lies in the same plane, $\mathbf{b}\times\mathbf{c}$ is also perpendicular to $\mathbf{a}$, *i.e.*,

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c})=0\ ;\ \text{or}\ [\mathbf{abc}]=0.$$

We can also argue in the following way :

Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar the volume of the parallelopiped formed by them is zero *i.e.*, $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=0$.

Important observations.

We remark here that $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=0$ suggests either of the following conclusions :

1. One of the three vectors is a zero vector.
2. Two of them are parallel.
3. Volume of the parallelopiped formed by them being zero, they are coplanar.

As to 2, we find that if **a** and **b** are parallel then $\mathbf{a}=n\mathbf{b}$ (where n is a scalar). Then

$$\begin{aligned}\mathbf{a}.\mathbf{b}\times\mathbf{c}&=n\mathbf{b}.\mathbf{b}\times\mathbf{c}\\&=n\mathbf{b}\times\mathbf{b}.\mathbf{c}\quad\text{(Interchange dot and cross)}\\&=\mathbf{0}.\mathbf{c}=0.\end{aligned}$$

Note that $[\mathbf{a}\ k\mathbf{a}\ \mathbf{c}]=0$, since $\mathbf{a}$, $k\mathbf{a}$ are parallel.

In particular, if two of them are equal the scalar triple product is *zero e.g.*, $[\mathbf{aac}]=0$.

3·42·1. Resolution of a vector r in the directions of three non-coplanar vectors a, b, c.

Any vector **r** may be resolved into component vectors in the directions of three *non-coplanar vectors* **a**, **b**, **c** as

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}.$$

We now proceed to find x, y, z.

Form the scalar product of each member with $\mathbf{b}\times\mathbf{c}$. Thus

$$\begin{aligned}\mathbf{r}.\mathbf{b}\times\mathbf{c}&=x\mathbf{a}.(\mathbf{b}\times\mathbf{c})+y\mathbf{b}.\mathbf{b}\times\mathbf{c}+z\mathbf{c}.\mathbf{b}\times\mathbf{c}\\&=x[\mathbf{abc}]\qquad\text{[other two scalar triple products vanish]}\end{aligned}$$

$$\therefore\quad x=\frac{\mathbf{r}.\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\quad(\because\ [\mathbf{abc}]\neq 0)$$

Similarly $y=\dfrac{\mathbf{r}.\mathbf{c}\times\mathbf{a}}{[\mathbf{bca}]}=\dfrac{[\mathbf{rca}]}{[\mathbf{abc}]}$; $z=\dfrac{\mathbf{r}.\mathbf{a}\times\mathbf{b}}{[\mathbf{cab}]}=\dfrac{[\mathbf{rab}]}{[\mathbf{abc}]}$.

Hence $\mathbf{r}=\dfrac{[\mathbf{rbc}]}{[\mathbf{abc}]}\mathbf{a}+\dfrac{[\mathbf{rca}]}{[\mathbf{abc}]}\mathbf{b}+\dfrac{[\mathbf{rab}]}{[\mathbf{abc}]}\mathbf{c}.$

3·43. Vector triple product.

We now consider the cross product of $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$

$$i.e., \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

We shall prove : $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$.

First note that in this expression the paranthesis is necessary, for $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ may not be equal to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. The sign of this product changes every time the order of the factors $\mathbf{a}$ and $(\mathbf{b} \times \mathbf{c})$ is changed in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ or whenever the order of the factors $\mathbf{b}$ and $\mathbf{c}$ is changed in $\mathbf{b} \times \mathbf{c}$.

Evidently, $\mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector perpendicular to both $\mathbf{a}$ and $(\mathbf{b} \times \mathbf{c})$.

$$i.e., \quad \mathbf{q}.\mathbf{a} = 0 \ ; \ \mathbf{q}.(\mathbf{b} \times \mathbf{c}) = 0 \text{ or } [\mathbf{qbc}] = 0.$$

From the second relation it follows that $\mathbf{q}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

[*Alternatively*, observe that $\mathbf{q}$ is perpendicular to $(\mathbf{b} \times \mathbf{c})$ which itself is a vector perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$ *i.e.*, $\mathbf{q}$ is perpendicular to the normal of the plane of $\mathbf{b}$ and $\mathbf{c}$ *i.e.*, $\mathbf{q}$ lies on the plane of $\mathbf{b}$ and $\mathbf{c}$.]

As $\mathbf{q}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$, we can express

$$\mathbf{q} = x\mathbf{b} + y\mathbf{c},$$

where x and y are suitable scalars.

Proof. To find the actual expression of $\mathbf{q}$ we consider a unit vector $\mathbf{j}$ along $\mathbf{b}$ and another unit vector $\mathbf{k}$ perpendicular to $\mathbf{j}$ in the plane of $\mathbf{b}$ and $\mathbf{c}$. Now we choose the unit vector $\mathbf{i}$ so that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right-handed vector-triad. We may now put

$$\mathbf{b} = 0\mathbf{i} + b_2\mathbf{j} + 0\mathbf{k} = b_2\mathbf{j}$$
$$\mathbf{c} = 0\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = c_2\mathbf{j} + c_3\mathbf{k}$$
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Evidently, $\mathbf{a}.\mathbf{b} = a_2 b_2$; $\mathbf{a}.\mathbf{c} = a_2 c_2 + a_3 c_3$;

$$\mathbf{b} \times \mathbf{c} = (b_2\mathbf{j}) \times (\mathbf{c}_2\mathbf{j} + c_3\mathbf{k}) = b_2 c_3 \mathbf{i} \quad (\because \ b_2\mathbf{j} \times c_2\mathbf{j} = \mathbf{0})$$

$$\text{Hence } \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_2c_3\mathbf{i})$$
$$= a_2b_2c_3\mathbf{j}\times\mathbf{i}+a_3b_2c_3\mathbf{k}\times\mathbf{i}$$
$$= -a_2b_2c_3\mathbf{k}+a_3b_2c_3\mathbf{j}$$
$$= (a_2c_2+a_3c_3)b_2\mathbf{j}-(a_2b_2)(c_2\mathbf{j}+c_3\mathbf{k})$$

(Introducing the term $a_2b_2c_2\mathbf{j}$)

$$= (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}.$$

Alternative Proof (Algebraic Standpoint).

To prove $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}$.

We suppose that three vectors are given in terms of number triples.

Let $\mathbf{a}=(a_1, a_2, a_3)$, $\mathbf{b}=(b_1, b_2, b_3)$, $\mathbf{c}=(c_1, c_2, c_2)$.

[We may otherwise suppose that the three vectors are expressed in terms of rectangular components :

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\ ;\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\ ;\ \mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}\ ].$$

Then $\mathbf{b}\times\mathbf{c}=[b_2c_3-b_3c_2,\ b_3c_1-b_1c_3,\ b_1c_2-b_2c_1]$.

$$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=[a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3),$$
$$a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1),$$
$$a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\ ] \quad \cdots \quad (1)$$

Again, $\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3=\Sigma a_1b_1$,

$$\mathbf{a}.\mathbf{c}=a_1c_1+a_2c_2+a_3c_3=\Sigma a_1c_1,$$
$$(\mathbf{a}.\mathbf{c})\mathbf{b}=[\ (\Sigma a_1c_1)b_1,\ (\Sigma a_1c_1)b_2,\ (\Sigma_1 a_1c_1)b_3\ ],$$
$$(\mathbf{a}.\mathbf{b})\mathbf{c}=[\ (\Sigma a_1b_1)c_1,\ (\Sigma a_1b_1)c_2,\ (\Sigma a_1b_1)c_3\ ].$$

Then, $(\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}$

$$=[b_1\Sigma a_1c_1-c_1\Sigma a_1b_1,\ b_2\Sigma a_1c_1-c_2\Sigma a_1b_1,\ b_3\Sigma a_1c_1-c_3\Sigma a_1b_1] \quad \cdots \quad (2)$$

Right-hand sides of (1) and (2) can be verified to be identical and so

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}.$$

Corollary 1. $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=(\mathbf{b}.\mathbf{a})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}=(\mathbf{a}.\mathbf{b})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}$.

2. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=-[\ (\mathbf{c}.\mathbf{b})\mathbf{a}-(\mathbf{c}.\mathbf{a})\mathbf{b}\ ]$

$$=(\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{b}.\mathbf{c})\mathbf{a}.$$

Rule to remember :

Vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *is to be expressed in terms of two vectors* $\mathbf{b}$ *and* $\mathbf{c}$ *in the bracket as* $x\mathbf{b}-y\mathbf{c}$ (x, y *are scalars*).

Both x and y are dot-products of two vectors. In each dot-product the outer vector **a** *will occur. The coefficient of* $\mathbf{b} = x = \mathbf{a}.\mathbf{c}$ *and the coefficient of* $\mathbf{c} = y = (\mathbf{a}.\mathbf{b})$.

If we call one vector in the paranthesis as adjacent to the vector outside, and the other remote, then

Vector triple product = (Outer . Remote) Adjacent − (Outer . Adjacent) Remote.

Illustrative Examples.

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Left side $= \{(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}\} + \{(\mathbf{b}.\mathbf{a})\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a}\} + \{(\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}\} = \mathbf{0}$.

2. If $\mathbf{a} = (-3, 7, 5)$, $\mathbf{b} = (-5, 7, -3)$, $\mathbf{c} = (7, -5, -3)$, find

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}),\ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}),\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

and verify the previous result.

Here
$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} = -71\mathbf{b} - 49\mathbf{c} \\ &= -71(-5, 7, -3) - 49(7, -5, -3) \\ &= (12, -252, 360) = 12(1, -21, 30). \end{aligned}$$

Similarly find the other two and verify the result in the previous illustration.

3·5. Products of four vectors.

The products of four or more vectors can be easily obtained by using the results already considered. We shall discuss a few important cases.

(A) Scalar product of four vectors.

We consider the dot-product of $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$ and prove the *Lagrange's identity* :

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) &= (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d}) - (\mathbf{a}.\mathbf{d})(\mathbf{b}.\mathbf{c}) \\ &= \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix} \end{aligned}$$

Proof. Obviously the dot-product of the two vectors $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ is a scalar. Let us call $\mathbf{q}=\mathbf{c}\times\mathbf{d}$. Then we have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b}).\mathbf{q}\\ &= \mathbf{a}.\mathbf{b}\times\mathbf{q} \text{ (interchange dot and cross)}\\ &= \mathbf{a}.\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\\ &= \mathbf{a}.\{(\mathbf{b}.\mathbf{d})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{d}\}\\ &= (\mathbf{b}.\mathbf{d})(\mathbf{a}.\mathbf{c})-(\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d})\\ &= \begin{vmatrix}\mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d}\\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d}\end{vmatrix}\end{aligned}$$

In particular, remember :

$$(\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})=(\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{b})-(\mathbf{a}.\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}.\mathbf{b})^2$$

(B) Vector product of four vectors.

We next consider the vector product of $(\mathbf{a}\times\mathbf{b})$ with $(\mathbf{c}\times\mathbf{d})$.

We shall prove

$$\left.\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{abd}]\mathbf{c}-[\mathbf{abc}]\mathbf{d}\\ &= [\mathbf{acd}]\mathbf{b}-[\mathbf{bcd}]\mathbf{a}\end{aligned}\right\} \quad \cdots \quad (1)$$

Proof. Obviously the cross-product of $(\mathbf{a}\times\mathbf{b})$ with $(\mathbf{c}\times\mathbf{d})$ is a vector. Let us call $\mathbf{p}=\mathbf{a}\times\mathbf{b}$ and $\mathbf{q}=\mathbf{c}\times\mathbf{d}$. Then we have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= \mathbf{p}\times(\mathbf{c}\times\mathbf{d})\\ &= (\mathbf{p}.\mathbf{d})\mathbf{c}-(\mathbf{p}.\mathbf{c})\mathbf{d}\\ &= \{(\mathbf{a}\times\mathbf{b}).\mathbf{d}\}\mathbf{c}-\{(\mathbf{a}\times\mathbf{b}).\mathbf{c}\}\mathbf{d}\\ &= [\mathbf{abd}]\mathbf{c}-[\mathbf{abc}]\mathbf{d} \qquad \cdots \quad (2)\end{aligned}$$

Again,

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})\times\mathbf{q}\\ &= -\mathbf{q}\times(\mathbf{a}\times\mathbf{b})\\ &= -\{(\mathbf{q}.\mathbf{b})\mathbf{a}-(\mathbf{q}.\mathbf{a})\mathbf{b}\}\\ &= (\mathbf{q}.\mathbf{a})\mathbf{b}-(\mathbf{q}.\mathbf{b})\mathbf{a}\\ &= \{\mathbf{c}\times\mathbf{d}.\mathbf{a}\}\mathbf{b}-\{\mathbf{c}\times\mathbf{d}.\mathbf{b}\}\mathbf{a}\\ &= [\mathbf{cda}]\mathbf{b}-[\mathbf{cdb}]\mathbf{a}\\ &= [\mathbf{acd}]\mathbf{b}-[\mathbf{bcd}]\mathbf{a}. \qquad \cdots \quad (3)\\ & \qquad \text{(by Parallelopiped law)}\end{aligned}$$

The relations (2) and (3) together give (1).

Corollary. It is evident from the relation (1) that

$$[\mathbf{abd}]\mathbf{c}-[\mathbf{abc}]\mathbf{d}=[\mathbf{acd}]\mathbf{b}-[\mathbf{bcd}]\mathbf{a}$$

i.e., $$[\mathbf{bcd}]\mathbf{a}-[\mathbf{acd}]\mathbf{b}+[\mathbf{abd}]\mathbf{c}-[\mathbf{abc}]\mathbf{d}=0,$$

which is a relation connecting four vectors **a**, **b**, **c**, **d**.

We may also write

$$\begin{aligned}[\mathbf{abc}]\mathbf{d}&=[\mathbf{bcd}]\mathbf{a}-[\mathbf{acd}]\mathbf{b}+[\mathbf{abd}]\mathbf{c}\\&=[\mathbf{bcd}]\mathbf{a}+[\mathbf{cad}]\mathbf{b}+[\mathbf{dab}]\mathbf{c}\quad(\because\ [\mathbf{cad}]=-[\mathbf{acd}]);\\&=[\mathbf{dbc}]\mathbf{a}+[\mathbf{dca}]\mathbf{b}+[\mathbf{dab}]\mathbf{c}.\end{aligned}$$

When **a**, **b**, **c** are non-coplanar, $[\mathbf{abc}]\neq 0$ and the formula

$$\mathbf{d}=\frac{[\mathbf{dbc}]}{[\mathbf{abc}]}\mathbf{a}+\frac{[\mathbf{dca}]}{[\mathbf{abc}]}\mathbf{b}+\frac{[\mathbf{dab}]}{[\mathbf{abc}]}\mathbf{c}\qquad\cdots\quad(4)$$

expresses any vector **d** as a linear combination of three non-coplanar vectors **a**, **b**, **c**.

More generally, any vector **r** can be expressed as a linear combination of three non-coplanar vectors **a**, **b**, **c** in the form

$$\mathbf{r}=\frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\mathbf{a}+\frac{[\mathbf{rca}]}{[\mathbf{abc}]}\mathbf{b}+\frac{[\mathbf{rab}]}{[\mathbf{abc}]}\mathbf{c}\qquad\cdots\quad(5)$$

which is a *very important and useful formula.*

3·51. A few Important Illustrations.

1. *Prove* $\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=(\mathbf{b}.\mathbf{d})\mathbf{a}\times\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}\times\mathbf{d}$.

Solution. Since $\mathbf{b}\times(\mathbf{c}\times\mathbf{d})=(\mathbf{b}.\mathbf{d})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{d}$, we obtain

$$\begin{aligned}\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}&=\mathbf{a}\times\{(\mathbf{b}.\mathbf{d})\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{d}\}\\&=(\mathbf{b}.\mathbf{d})\mathbf{a}\times\mathbf{c}-(\mathbf{b}.\mathbf{c})\mathbf{a}\times\mathbf{d}\end{aligned}$$

2. *Expand* $\mathbf{a}\times[\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}]$ *in the form*

$$\{(\mathbf{a}.\mathbf{d})(\mathbf{c}.\mathbf{e})-(\mathbf{c}.\mathbf{d})(\mathbf{a}.\mathbf{e})\}\mathbf{b}+\{(\mathbf{c}.\mathbf{d})\mathbf{e}-(\mathbf{c}.\mathbf{e})\mathbf{d}\}(\mathbf{a}.\mathbf{b})$$

This is left as an exercise.

3. *Show that* $[\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}]=[\mathbf{abc}]^2$.

Solution. Let us put $\mathbf{p}=\mathbf{b}\times\mathbf{c}$, $\mathbf{q}=\mathbf{c}\times\mathbf{a}$, $\mathbf{r}=\mathbf{a}\times\mathbf{b}$.

Then the left-hand side $= [\mathbf{pqr}] = \mathbf{p}.\mathbf{q} \times \mathbf{r}$

$$= \mathbf{p}.\mathbf{q} \times (\mathbf{a} \times \mathbf{b}) \qquad [\text{ putting } \mathbf{r} = \mathbf{a} \times \mathbf{b}\ ]$$

$$= \mathbf{p}.\{(\mathbf{q}.\mathbf{b})\mathbf{a} - (\mathbf{q}.\mathbf{a})\mathbf{b}\} = \mathbf{p}.\{[\mathbf{abc}]\mathbf{a}\}$$

$$(\because\ \mathbf{q}.\mathbf{b} = \mathbf{c} \times \mathbf{a}.\mathbf{b} = [\mathbf{abc}]\ ;\ \mathbf{q}.\mathbf{a} = \mathbf{c} \times \mathbf{a}.\mathbf{a} = [\mathbf{caa}] = 0\ )$$

$$= [\mathbf{abc}](\mathbf{p}.\mathbf{a})$$

$$= [\mathbf{abc}](\mathbf{b} \times \mathbf{c}.\mathbf{a}) = [\mathbf{abc}]^2$$

4. *Reduce the expression* $(\mathbf{b} + \mathbf{c}).\{(\mathbf{c} + \mathbf{a}) \times (\mathbf{a} + \mathbf{b})\}$ *in its simplest form and prove that it vanishes when* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are coplanar.*

Solution. We have $(\mathbf{c} + \mathbf{a}) \times (\mathbf{a} + \mathbf{b})$

$$= \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b} + \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$$

$$= \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} \quad (\because\ \mathbf{a} \times \mathbf{a} = \mathbf{0}\ )$$

Hence $(\mathbf{b} + \mathbf{c}).\ \{(\mathbf{c} + \mathbf{a}) \times (\mathbf{a} + \mathbf{b})\}$

$$= (\mathbf{b} + \mathbf{c}).\{\mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b}\}$$

$$= (\mathbf{b}.\mathbf{c} \times \mathbf{a} - \mathbf{b}.\mathbf{b} \times \mathbf{c} + \mathbf{b}.\mathbf{a} \times \mathbf{b} + \mathbf{c}.\mathbf{c} \times \mathbf{a} - \mathbf{c}.\mathbf{b} \times \mathbf{c} + \mathbf{c}.\mathbf{a} \times \mathbf{b}$$

$$= [\mathbf{bca}] + [\mathbf{cab}] \qquad (\because\ \text{scalar triple product vanishes when two vectors are equal})$$

$$= 2[\mathbf{abc}]$$

$= \mathbf{0}$, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar.

5. *Prove* $[\mathbf{a} \times \mathbf{b},\ \mathbf{c} \times \mathbf{d},\ \mathbf{e} \times \mathbf{f}] = [\mathbf{abd}][\mathbf{cef}] - [\mathbf{abc}][\mathbf{def}]$

$$= [\mathbf{abe}][\mathbf{fcd}] - [\mathbf{abf}][\mathbf{ecd}]$$

$$= [\mathbf{cda}][\mathbf{bef}] - [\mathbf{cdb}][\mathbf{aef}]$$

Solution. Let us put $\mathbf{p} = \mathbf{a} \times \mathbf{b}$, $\mathbf{q} = \mathbf{c} \times \mathbf{d}$, $\mathbf{r} = \mathbf{e} \times \mathbf{f}$.

Then we have $[\mathbf{pqr}] = \mathbf{p}.\mathbf{q} \times \mathbf{r} = \mathbf{q}.\mathbf{r} \times \mathbf{p} = \mathbf{r}.\mathbf{p} \times \mathbf{q}$

$$\mathbf{q} \times \mathbf{r} = (\mathbf{c} \times \mathbf{d}) \times (\mathbf{e} \times \mathbf{f}) = [\mathbf{cef}]\mathbf{d} - [\mathbf{def}]\mathbf{c}$$

Hence $\mathbf{p}.\mathbf{q} \times \mathbf{r} = [\mathbf{cef}]\mathbf{p}.\mathbf{d} - [\mathbf{def}]\mathbf{p}.\mathbf{c}$

$$= [\mathbf{cef}][\mathbf{abd}] - [\mathbf{def}][\mathbf{abc}]$$

Now $\mathbf{q}.\mathbf{r} \times \mathbf{p}$ and $\mathbf{r}.\mathbf{p} \times \mathbf{q}$ will give other two relations.

6. *Prove* $(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = 0.$

Solution. We have : $(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) = (\mathbf{b}.\mathbf{a})(\mathbf{c}.\mathbf{d}) - (\mathbf{c}.\mathbf{a})(\mathbf{b}.\mathbf{d})$

$$(\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) = (\mathbf{c}.\mathbf{b})(\mathbf{a}.\mathbf{d}) - (\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{d})$$

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d}) - (\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d})$$

Their sum is evidently zero.

7. *Prove*

$$(\mathbf{b}\times\mathbf{c})\times(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=-2[\mathbf{abc}]\mathbf{d}$$

This is left as an exercise; compare this result with that of the previous illustration.

8. *Prove that*

$$[\mathbf{pqr}]\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b} & \mathbf{p}\\ \mathbf{q.a} & \mathbf{q.b} & \mathbf{q}\\ \mathbf{r.a} & \mathbf{r.b} & \mathbf{r}\end{vmatrix}$$

where **p**, **q**, **r** *are any three non-coplanar vectors.*

Solution. Since **p**, **q**, **r** are a set of non-coplanar vectors, we have $[\mathbf{pqr}]\neq 0$. Consider $\mathbf{a}\times\mathbf{b}=\mathbf{d}$. Then the four vectors **p**, **q**, **r**, **d** can be connected by the linear relation

$$\mathbf{d}=\frac{\mathbf{d.q}\times\mathbf{r}}{[\mathbf{pqr}]}\mathbf{p}+\frac{\mathbf{d.r}\times\mathbf{p}}{[\mathbf{pqr}]}\mathbf{q}+\frac{\mathbf{d.p}\times\mathbf{q}}{[\mathbf{pqr}]}\mathbf{r}. \qquad \cdots \quad (1)$$

Now observe

$$\mathbf{d.q}\times\mathbf{r}=(\mathbf{a}\times\mathbf{b}).(\mathbf{q}\times\mathbf{r})=(\mathbf{q}\times\mathbf{r}).(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\mathbf{q.a} & \mathbf{q.b}\\ \mathbf{r.a} & \mathbf{r.b}\end{vmatrix};$$

$$\mathbf{d.r}\times\mathbf{p}=(\mathbf{a}\times\mathbf{b}).(\mathbf{r}\times\mathbf{p})=-(\mathbf{p}\times\mathbf{r}).(\mathbf{a}\times\mathbf{b})=-\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b}\\ \mathbf{r.a} & \mathbf{r.b}\end{vmatrix};$$

and $\mathbf{d.p}\times\mathbf{q}=(\mathbf{a}\times\mathbf{b}).(\mathbf{p}\times\mathbf{q})=(\mathbf{p}\times\mathbf{q}).(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b}\\ \mathbf{q.a} & \mathbf{q.b}\end{vmatrix}.$

Then (1) reduces to

$$[\mathbf{pqr}]\mathbf{d}=\begin{vmatrix}\mathbf{q.a} & \mathbf{q.b}\\ \mathbf{r.a} & \mathbf{r.b}\end{vmatrix}\mathbf{p}-\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b}\\ \mathbf{r.a} & \mathbf{r.b}\end{vmatrix}\mathbf{q}+\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b}\\ \mathbf{q.a} & \mathbf{q.b}\end{vmatrix}\mathbf{r}$$

$$=\begin{vmatrix}\mathbf{p.a} & \mathbf{p.b} & \mathbf{p}\\ \mathbf{q.a} & \mathbf{q.b} & \mathbf{q}\\ \mathbf{r.a} & \mathbf{r.b} & \mathbf{r}\end{vmatrix}$$

9. *Prove that* $[\mathbf{pqr}][\mathbf{abc}] = \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} & \mathbf{p.c} \\ \mathbf{q.a} & \mathbf{q.b} & \mathbf{q.c} \\ \mathbf{r.a} & \mathbf{r.b} & \mathbf{r.c} \end{vmatrix}$

Hence deduce $[\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a.a} & \mathbf{a.b} & \mathbf{a.c} \\ \mathbf{b.a} & \mathbf{b.b} & \mathbf{b.c} \\ \mathbf{c.a} & \mathbf{c.b} & \mathbf{c.c} \end{vmatrix}$

Solution. Writing $\mathbf{a} \times \mathbf{b} = \mathbf{d}$ and considering four vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{d}$ we obtain the linear relation connecting these four vectors. Thus

$$[\mathbf{pqr}]\mathbf{d} = \begin{vmatrix} \mathbf{q.a} & \mathbf{q.b} \\ \mathbf{r.a} & \mathbf{r.b} \end{vmatrix} \mathbf{p} - \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} \\ \mathbf{r.a} & \mathbf{r.b} \end{vmatrix} \mathbf{q} + \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} \\ \mathbf{q.a} & \mathbf{q.b} \end{vmatrix} \mathbf{r}.$$

Multiplying *scalarly* with $\mathbf{c}$, we obtain

$$[\mathbf{pqr}]\mathbf{d.c} = \begin{vmatrix} \mathbf{q.a} & \mathbf{q \cdot b} \\ \mathbf{r.a} & \mathbf{r.b} \end{vmatrix} \mathbf{p.c} - \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} \\ \mathbf{r.a} & \mathbf{r.b} \end{vmatrix} \mathbf{q.c} + \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} \\ \mathbf{q.a} & \mathbf{q.b} \end{vmatrix} \mathbf{r.c}$$

i.e. $[\mathbf{pqr}][\mathbf{abc}] = \begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} & \mathbf{p.c} \\ \mathbf{q.a} & \mathbf{q.b} & \mathbf{q.c} \\ \mathbf{r.a} & \mathbf{r.b} & \mathbf{r.c} \end{vmatrix}$

Note that $\mathbf{d.c} = \mathbf{a} \times \mathbf{b.c} = \mathbf{a.b} \times \mathbf{c} = [\mathbf{abc}]$.

Writing $\mathbf{p} = \mathbf{a}$, $\mathbf{q} = \mathbf{b}$, $\mathbf{r} = \mathbf{c}$, we get the relation for $[\mathbf{abc}]^2$.

10. *Prove that* $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *when and only when* $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

Solution. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$,

if and only if $(\mathbf{a.c})\mathbf{b} - (\mathbf{b.c})\mathbf{a} = (\mathbf{a.c})\mathbf{b} - (\mathbf{a.b})\mathbf{c}$

i.e., if and only if $(\mathbf{b.c})\mathbf{a} = (\mathbf{a.b})\mathbf{c}$ $\cdots$ (1)

This is possible if either

(i) one of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a zero vector,

or if (ii) $\mathbf{b}$ is perpendicular to $\mathbf{a}$ as well as $\mathbf{c}$,

or if (iii) $\mathbf{c}$ is parallel (or collinear) with $\mathbf{a}$.

All these three conditions are implied by the equation

$$(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0} \qquad \cdots \quad (2)$$

Observe that $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = (\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ and so (1) implies (2).

11. *Prove that*

$$2\,(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} -\mathbf{a} & -\mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$; $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$;

$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ *and* $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$.

Solution. We have

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}$$

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \mathbf{c} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \mathbf{d}$$

Further, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a}$

$$= \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} \mathbf{b} - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \mathbf{a}$$

Now add the two expressions for $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$; the required result follows.

3·52. Reciprocal system of vectors.

DEFINITION. Consider a set of three *non-coplanar* vectors **a, b, c**. *All* vectors can be expressed linearly in terms of them. We say that the set **a, b, c** form a *basis* for space of three dimensions.

Note that if **a, b, c** form a basis then $[\mathbf{abc}] \neq 0$ (why ?).

Next consider another basis $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ such that

$$\left.\begin{array}{lll} \mathbf{a}.\mathbf{a}'=1, & \mathbf{a}.\mathbf{b}'=0, & \mathbf{a}.\mathbf{c}'=0 \\ \mathbf{b}.\mathbf{a}'=0, & \mathbf{b}.\mathbf{b}'=1, & \mathbf{b}.\mathbf{c}'=0 \\ \mathbf{c}.\mathbf{a}'=0, & \mathbf{c}.\mathbf{b}'=0, & \mathbf{c}.\mathbf{c}'=1 \end{array}\right\} \cdots (1)$$

We then say that the set $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ is *reciprocal* to the set $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Expressions for $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Consider the three relations in the first column of (1). The relations $\mathbf{b}.\mathbf{a}'=\mathbf{c}.\mathbf{a}'=0$ imply that $\mathbf{a}'$ is perpendicular to $\mathbf{b}$ as well as $\mathbf{c}$ and hence parallel to $\mathbf{b}\times\mathbf{c}$. In other words, $\mathbf{a}'=k\ \mathbf{b}\times\mathbf{c}$, where k is a number to be determined. Further from $\mathbf{a}.\mathbf{a}'=1$ we derive $k=1/[\mathbf{abc}]$. We thus obtain,

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}, \qquad \cdots (2)$$

$\mathbf{b}'$ and $\mathbf{c}'$ being derived in a similar manner from the relations of second and third column respectively of (1).

The symmetry of the relations in (1) also give

$$\mathbf{a}=\frac{\mathbf{b}'\times\mathbf{c}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{b}=\frac{\mathbf{c}'\times\mathbf{a}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{c}=\frac{\mathbf{a}'\times\mathbf{b}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']}. \cdots (3)$$

We may thus call the set $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ reciprocal to $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$.

Relations (1) derived from relations (2).

We next suppose that $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are defined in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ by the relations (2). We then proceed to derive the relations (1). Observe that

$$\mathbf{a}.\mathbf{a}'=\mathbf{a}.\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\frac{[\mathbf{abc}]}{[\mathbf{abc}]}=1$$

and $$\mathbf{a}.\mathbf{b}'=\mathbf{a}.\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}=\frac{[\mathbf{aca}]}{[\mathbf{abc}]}=0 \qquad (\because\ [\mathbf{aca}]=0)$$

Similarly other relations of (1) can be derived.

Our next step is to obtain relations (3), $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ being defined by (2).

Now $$\mathbf{b}'\times\mathbf{c}'=\frac{(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})}{[\mathbf{abc}]^2}=\frac{[\mathbf{cab}]}{[\mathbf{abc}]^2}\,\mathbf{a}=\frac{\mathbf{a}}{[\mathbf{abc}]} \qquad (\text{art. } 3{\cdot}5)$$

whence $$\mathbf{a}'.\mathbf{b}'\times\mathbf{c}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}.\frac{\mathbf{a}}{[\mathbf{abc}]}=\frac{1}{[\mathbf{abc}]}.$$

This leads to $[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}']=1$ ... (4)

Consequently. $\mathbf{a}=\dfrac{\mathbf{b}'\times\mathbf{c}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']}$.

Similarly obtain the expressions for $\mathbf{b}$ and $\mathbf{c}$ as given in (3).

Incidentally we have shown that $[\mathbf{abc}]$ is reciprocal to $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$; this gives further justification for the name *reciprocal* to the *sets*. The relation (1) also implies that the box-products $[\mathbf{abc}]$ and $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$ must have the same sign *i.e.*, *two sets which are reciprocal to one another are either both right-handed or both left-handed.*

Other relations connecting two reciprocal sets.

1. $\mathbf{a}\times\mathbf{a}'+\mathbf{b}\times\mathbf{b}'+\mathbf{c}\times\mathbf{c}'=0.$

This follows from the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$$

if we divide throughout by $[\mathbf{abc}]$.

2. Any vector $\mathbf{r}$ may be expressed as

$$\mathbf{r}=\frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\,\mathbf{a}+\frac{[\mathbf{rca}]}{[\mathbf{abc}]}\,\mathbf{b}+\frac{[\mathbf{rab}]}{[\mathbf{abc}]}\,\mathbf{c}. \qquad (\text{art. } 3{\cdot}5 \text{ Cor.})$$

This may be written as

$$\mathbf{r}=(\mathbf{r}.\mathbf{a}')\,\mathbf{a}+(\mathbf{r}.\mathbf{b}')\,\mathbf{b}+(\mathbf{r}.\mathbf{c}')\,\mathbf{c}$$

and from symmetry it also follows that any vectoı $\mathbf{r}$ can be expressed as

$$\mathbf{r}=(\mathbf{r}.\mathbf{a})\,\mathbf{a}'+(\mathbf{r}.\mathbf{b})\,\mathbf{b}'+(\mathbf{r}.\mathbf{c})\,\mathbf{c}'.$$

3. When a basis and its reciprocal are identical, say

$$\mathbf{a}=\mathbf{a}', \ \mathbf{b}=\mathbf{b}', \ \mathbf{c}=\mathbf{c}'$$

then the relations (1) give

$$\mathbf{a}.\mathbf{a}=\mathbf{b}.\mathbf{b}=\mathbf{c}.\mathbf{c}=1, \ \mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{c}=\mathbf{c}.\mathbf{a}=0.$$

These equations characterise an *orthogonal type of unit vectors.*

Hence a basis is self-reciprocal if and only if it consists of a mutually orthogonal type of unit vectors. Our familiar set of unit vectors **i, j, k** is self-orthogonal and $[\mathbf{ijk}]=1$.

Note. We give a short discussion on *General bases* :

With an arbitrary basis **a, b, c** we may express any vector **r** in the form $\mathbf{r}=r_1\mathbf{a}+r_2\mathbf{b}+r_3\mathbf{c}$ where $r_1=\mathbf{r}.\mathbf{a}'$, $r_2=\mathbf{r}.\mathbf{b}'$ and $r_3=\mathbf{r}.\mathbf{c}'$ where the basis $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ is reciprocal to **a, b, c.** We call (r_1, r_2, r_3) components of **r** for the basis **a, b, c.**

We agree to take (r_1', r_2', r_3') as components of **r** for the reciprocal basis $\mathbf{a}', \mathbf{b}', \mathbf{c}'$. We proceed to find an expression for $\mathbf{q}\times\mathbf{r}$; we take both vectors to the same basis $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, then

$$\mathbf{q}\times\mathbf{r}=(q_1'\mathbf{a}'+q'_2\mathbf{b}'+q_3'\mathbf{c}')\times(r_1'\mathbf{a}'+r'_2\mathbf{b}'+r'_3\mathbf{c}')$$

$$=\begin{vmatrix} q_2' & q_3' \\ r_2' & r_3' \end{vmatrix}\mathbf{b}'\times\mathbf{c}'+\begin{vmatrix} q'_3 & q'_1 \\ r_3' & r'_1 \end{vmatrix}\mathbf{c}'\times\mathbf{a}'+\begin{vmatrix} q_1' & q_2' \\ r_1' & r_2' \end{vmatrix}\mathbf{a}'\times\mathbf{b}'.$$

But we know $\mathbf{b}'\times\mathbf{c}'=\mathbf{a}/[\mathbf{abc}]$; etc. and hence

$$\mathbf{q}\times\mathbf{r}=\frac{1}{[\mathbf{abc}]}\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ q_1' & q_2' & q_3' \\ r_1' & r_2' & r_3' \end{vmatrix}$$

whence $$\mathbf{p}.\mathbf{q}\times\mathbf{r}=\frac{1}{[\mathbf{abc}]}\begin{vmatrix} p_1' & p_2' & p_3' \\ q_1' & q_2' & q_3' \\ r_1' & r_2' & r_3' \end{vmatrix}=\frac{1}{[\mathbf{abc}]}\begin{vmatrix} \mathbf{p}.\mathbf{a} & \mathbf{p}.\mathbf{b} & \mathbf{p}.\mathbf{c} \\ \mathbf{q}.\mathbf{a} & \mathbf{q}.\mathbf{b} & \mathbf{q}.\mathbf{c} \\ \mathbf{r}.\mathbf{a} & \mathbf{r}.\mathbf{b} & \mathbf{r}.\mathbf{c} \end{vmatrix}$$

This yields the formula $$[\mathbf{pqr}][\mathbf{abc}]=\begin{vmatrix} \mathbf{p}.\mathbf{a} & \mathbf{p}.\mathbf{b} & \mathbf{p}.\mathbf{c} \\ \mathbf{q}.\mathbf{a} & \mathbf{q}.\mathbf{b} & \mathbf{q}.\mathbf{c} \\ \mathbf{r}.\mathbf{a} & \mathbf{r}.\mathbf{b} & \mathbf{r}.\mathbf{c} \end{vmatrix}$$ (Illustration 9 ; art 3·51).

If we refer **a, b, c** and **p, q, r** to the basis **i, j, k**, the left member becomes the product of determinants :

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and the formula given above is equivalent to *row by row* rule for multiplication of two determinants.

Examples. III(C)

1. Calculate simplest values of

$$\mathbf{a}.\mathbf{b},\ \mathbf{a}\times\mathbf{b},\ \mathbf{c}.\mathbf{a}\times\mathbf{b}$$

where **a, b, c** are defined by

$$\mathbf{a}=5\mathbf{i}-4\mathbf{j}+\mathbf{k}\ ;\ \mathbf{b}=-4\mathbf{i}+3\mathbf{j}-2\mathbf{k}\ ;\ \mathbf{c}=\mathbf{i}-2\mathbf{j}-7\mathbf{k},$$

(**i, j, k** are mutually perpendicular right-handed system of vectors).

2. Explain fully the difference between the geometrical implications of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ where **a, b, c** are any three non-coplanar vectors.

3. If $\mathbf{a}=-3\mathbf{i}+7\mathbf{j}+5\mathbf{k}$; $\mathbf{b}=-5\mathbf{i}+7\mathbf{j}-3\mathbf{k}$; $\mathbf{c}=7\mathbf{i}-5\mathbf{j}-3\mathbf{k}$, calculate the scalar magnitude of the volume of the parallelopiped having **a, b, c** as adjacent sides with a common origin and also find the expressions for three cross-products $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$.

4. Show that the vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ are coplanar.

5. If **p, q, r** denote the vectors $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$ respectively show that **a** is parallel to $\mathbf{q}\times\mathbf{r}$, **b** is parallel to $\mathbf{r}\times\mathbf{p}$, **c** is parallel to $\mathbf{p}\times\mathbf{q}$.

6. Prove that

(i) $\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$;

(ii) $(\mathbf{a} - \mathbf{d}).(\mathbf{b} - \mathbf{c}) + (\mathbf{b} - \mathbf{d}).(\mathbf{c} - \mathbf{a}) + (\mathbf{c} - \mathbf{d}).(\mathbf{a} - \mathbf{b}) = 0$;

(iii) $(\mathbf{a} - \mathbf{d}) \times (\mathbf{b} - \mathbf{c}) + (\mathbf{b} - \mathbf{d}) \times (\mathbf{c} - \mathbf{a}) + (\mathbf{c} - \mathbf{d}) \times (\mathbf{a} - \mathbf{b})$
$= 2(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$.

7. Find the set of vectors reciprocal to

$$\mathbf{a} = (1, 0, 0),\ \mathbf{b} = (1, 1, 0),\ \mathbf{c} = (1, 1, 1).$$

Find the components of $\mathbf{r} = (2, 3, 4)$ relative to the bases $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

8. Prove the formula for the area of a plane triangle with sides a, b, c :

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \tfrac{1}{2}(a+b+c).$$

9. Find the set reciprocal to $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$.

10. Find two vectors $\mathbf{a}$, $\mathbf{b}$ which are of equal magnitude, are mutually perpendicular, each have the x-component 5, and are each perpendicular to $3\mathbf{j} + 4\mathbf{k}$.

11. Express a vector $\mathbf{a}$ as the sum of two component vectors, one parallel and the other perpendicular to the vector $\mathbf{b}$ in the form :

$$\mathbf{a} = \frac{1}{b^2}\{(\mathbf{a}.\mathbf{b})\,\mathbf{b} + \mathbf{b} \times (\mathbf{a} \times \mathbf{b})\}.$$

12. Spherical Trigonometry.

If three points A, B, C on the surface of a sphere are joined by arcs of great circles, the figure formed is called a *spherical triangle ABC*. We consider a sphere of unit radius with its centre at O (draw a figure). The position vectors of A, B, C are respectively

$$\overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{b},\ \overrightarrow{OC} = \mathbf{c}$$

relative to the centre O ; $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are then three unit vectors ; clearly $[\mathbf{abc}] > 0$. Let a, b, c denote the sides (arcs of great circles) opposite to the *interior* (dihedral) angles A, B, C. These sides are, in fact, angles BOC, COA, AOB which these arcs subtend at the centre and the angle A is the angle between the planes AOB and AOC ; other angles are similarly defined. Let all the angles and sides are less than π. Now

$$\mathbf{b}.\mathbf{c} = \cos a, \quad \mathbf{c}.\mathbf{a} = \cos b, \quad \mathbf{a}.\mathbf{b} = \cos c ;$$

$$\mathbf{b} \times \mathbf{c} = \sin a\ \mathbf{a}', \ \mathbf{c} \times \mathbf{a} = \sin b\ \mathbf{b}', \ \mathbf{a} \times \mathbf{b} = \sin c\ \mathbf{c}'.$$

Here $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are unit vectors normal to the planes BOC, COA, AOB and they include angles

$(\mathbf{b}', \mathbf{c}') = \pi - A = a'$ (say) ; $(\mathbf{c}', \mathbf{a}') = \pi - B = b'$; $(\mathbf{a}', \mathbf{b}') = \pi - C = c'$,

so that a', b', c' are the *exterior* dihedral angles at A, B, C. Moreover

$$\mathbf{b}' \times \mathbf{c}' = \frac{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})}{\sin b \sin c} = \frac{[\mathbf{abc}]}{\sin b \sin c}\mathbf{a},$$

a positive multiple of $\mathbf{a}$. Thus we have the equations

$$\mathbf{b}'.\mathbf{c}' = \cos a', \quad \mathbf{c}'.\mathbf{a}' = \cos b'. \quad \mathbf{a}'.\mathbf{b}' = \cos c'$$

$$\mathbf{b}' \times \mathbf{c}' = \sin a' \mathbf{a}, \quad \mathbf{c}' \times \mathbf{a}' = \sin b' \mathbf{b}, \quad \mathbf{a}' \times \mathbf{b}' = \sin c' \mathbf{c}$$

$$[\mathbf{abc}] = \sin a\ \mathbf{a}.\mathbf{a}' = \sin b\ \mathbf{b}.\mathbf{b}' = \sin c\ \mathbf{c}.\mathbf{c}'$$

$$[\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \sin a'\ \mathbf{a}.\mathbf{a}' = \sin b'\ \mathbf{b}.\mathbf{b}' = \sin c'\ \mathbf{c}.\mathbf{c}'$$

Hence, on division,

$$\frac{\sin a}{\sin a'} = \frac{\sin b}{\sin b'} = \frac{\sin c}{\sin c'}$$

$$\text{or,} \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \quad (\textit{ Sine Law })$$

Again from, $(\mathbf{c} \times \mathbf{a}).(\mathbf{a} \times \mathbf{b}) = (\mathbf{c}.\mathbf{a})(\mathbf{a}.\mathbf{b}) - (\mathbf{b}.\mathbf{c})$, we derive

$$\sin b \sin c\ \mathbf{b}'.\mathbf{c}' = \cos b \cos c - \cos a$$

or $-\sin b \sin c \cos A = \cos b.\cos c - \cos a \quad [\because\ \mathbf{b}'.\mathbf{c}' = \cos(\pi - A)]$

whence, $\cos a = \cos b \cos c + \sin b \sin c \cos A$ (*Cosine Law*)

Cyclic permutation gives the cosine law for $\cos B$ and $\cos C$.

Hints and Answers

1. $\mathbf{a.b} = -34$, $\mathbf{a} \times \mathbf{b} = 5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$, $\mathbf{c.a} \times \mathbf{b} = 0$.

3. Volume $= 264$ units ; $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 12\,(\mathbf{i} - 21\mathbf{j} + 30\mathbf{k})$; etc.

4. Their sum is a zero vector *i.e.*, there exists a linear relation connecting them ; hence etc.

5. $(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = [\mathbf{cab}]\mathbf{a} - [\mathbf{aab}]\mathbf{c} = [\mathbf{abc}]\mathbf{a}$; hence etc.

7. $[\mathbf{abc}] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$; $\mathbf{b} \times \mathbf{c} = (1, -1, 0)$; $\mathbf{a}' = \dfrac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} = (1, -1, 0)$.

Similarly obtain $\mathbf{b}'$, $\mathbf{c}'$. Relative to the base $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the components are $\mathbf{r.a}'$, $\mathbf{r.b}'$, $\mathbf{r.c}'$; see that $\mathbf{r.a}' = (2, 3, 4).(1, -1, 0) = -1$.

8. From $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, deduce $c^2 = a^2 + b^2 + 2\mathbf{a.b}$; now

$$4\triangle^2 = (\mathbf{a} \times \mathbf{b}).(\mathbf{a} \times \mathbf{b}) = a^2 b^2 - (\mathbf{a.b})^2 = (ab + \mathbf{a.b})(ab - \mathbf{a.b}) \text{ ; now proceed.}$$

9. $\dfrac{\mathbf{b} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|^2}, \dfrac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}}{|\mathbf{a} \times \mathbf{b}|^2}, \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$.

10. $5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

11. $\mathbf{a}.\hat{\mathbf{b}} = a \cos\theta =$ component of $\mathbf{a}$ in the direction of $\mathbf{b}$; also $b\hat{\mathbf{b}} = \mathbf{b}$.

$\therefore$ Component parallel to $\mathbf{b}$ is $(\mathbf{a}.\hat{\mathbf{b}})\,\hat{\mathbf{b}} = (1/b^2)\,(\mathbf{a.b})\mathbf{b}$ and so the perpendicular component is $\mathbf{a} - \dfrac{1}{b^2}\,(\mathbf{a.b})\mathbf{b} = \dfrac{1}{b^2}\,\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

Summary of Chapter 3.

1. Scalar product (or Dot-product) of two vectors a and b.

A. *Definition* : $\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ where θ is the smallest angle between $\mathbf{a}$ and $\mathbf{b}$. Thus $\mathbf{a.b}$ is a number which for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, is positive, zero or negative according as the angle θ is acute, right or obtuse.

B. *Condition of perpendicularity* : For proper vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a.b} = 0$ implies $\mathbf{a} \perp \mathbf{b}$. In particular, $\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$.

C. *Dot-product obeys*

Commutative Law : $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

Associative Law (with *scalars*) ; $\mathbf{a}.m\mathbf{b} = (m\mathbf{a}).\mathbf{b} = m(\mathbf{a}.\mathbf{b})$.

Associative Law (with *vectors*) : need not be considered since $(\mathbf{a}.\mathbf{b}).\mathbf{c}$ is not defined.

Note that in general $(\mathbf{a}.\mathbf{b})\,\mathbf{c} \neq \mathbf{a}\,(\mathbf{b}.\mathbf{c})$, though both are defined.

Distributive Law : $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.

D. *Parallel vectors* : If **a** and **b** are parallel then $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|$; in particular $\mathbf{a}.\mathbf{a} = \mathbf{a}^2 = |\mathbf{a}|^2$; also $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$.

E. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

F. *Angle between two vectors* :

$$\theta = \cos^{-1}\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \cos^{-1}\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

2. Vector product (or Cross product) of two vectors a and b.

A. *Definition.* $\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\varepsilon$,
where ε is a unit vector ($|\varepsilon| = 1$) and ε is $\perp$ **a** and **b** but the sense of ε is such that **a**, **b**, ε form a right-handed system of vector triads. Thus $\mathbf{a}\times\mathbf{b}$ is a well-defined vector.

B. *Vanishing of* $\mathbf{a}\times\mathbf{b}$. $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ implies either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or both **a** and **b** are zero vectors or **a** and **b** are parallel.

For two proper vectors **a** and **b** the relation $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ implies $\mathbf{a} \parallel \mathbf{b}$.

In particular, $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$.

C. *Vector product is anti commutative* : $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$.

D. Definition implies $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ form a right-handed triad of vectors.

We have the useful relations :

$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$.

E. *Associativity with scalars* :

$(m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = m\,(\mathbf{a} \times \mathbf{b})$;

$(m\mathbf{a}) \times (n\mathbf{b}) = mn\,(\mathbf{a} \times \mathbf{b}) = (mn\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (mn\mathbf{b})$.

But vector product does not obey the associative law when associated with vectors ; in general $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

F. *Vector product is distributive with respect to addition.*

$$\left.\begin{array}{l} \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\ (\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \end{array}\right\}$$

G. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

H. $\mathbf{a} \times \mathbf{b}$ = vector area of the parallelogram whose sides are $\mathbf{a}$ and $\mathbf{b}$.

I. If $\mathbf{c} \neq \mathbf{0}$ then from $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ or $(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0}$ we can conclude either that $\mathbf{a} - \mathbf{b} = \mathbf{0}$ or that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are parallel $(\mathbf{a} = \mathbf{b} + k\mathbf{c})$. But we cannot *cancel* $\mathbf{c}$ to obtain $\mathbf{a} = \mathbf{b}$ unless we know that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are *not parallel*.

3. Product of three vectors a, b, c.

A. *Scalar triple product.* $\mathbf{a} . (\mathbf{b} \times \mathbf{c}) = [\mathbf{abc}]$ is numerically equal to the volume of a parallelopiped having $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as concurrent edges. Its sign is positive or negative according as $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a right-handed or left-handed system of vector-triads. Hence the name *box-product*.

Parallelopiped law :

(i) $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}]$;

(ii) Dot and cross may be interchanged :

$$[\mathbf{abc}] = \mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c}.$$

(iii) For every change of cyclical order a negative sign is introduced : $\mathbf{a}.\mathbf{b} \times \mathbf{c} = -\mathbf{a}.\mathbf{c} \times \mathbf{b}$.

Coplanarity : $[\mathbf{abc}] = 0$ implies **a**, **b** and **c** are coplanar if the vectors **a**, **b** and **c** are non-zero vectors.

B. *Vector triple product.* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\, \mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

Triple product = (Outer. Remote) Adjacent − (Outer. Adjacent) Remote

We have $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ if and only if $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

An useful identity : $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

4. Product of four vectors :

Scalar product : $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix}$

Vector product : $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\, \mathbf{b} - [\mathbf{bcd}]\, \mathbf{a}$
$= [\mathbf{abd}]\, \mathbf{c} - [\mathbf{abc}]\, \mathbf{d}$

Any vector **r** can be expressed as

$$\mathbf{r} = \frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\, \mathbf{a} + \frac{[\mathbf{rca}]}{[\mathbf{abc}]}\, \mathbf{b} + \frac{[\mathbf{rab}]}{[\mathbf{abc}]}\, \mathbf{c},$$

where **a**, **b**, **c** are any three non-coplanar vectors *i.e.* $[\mathbf{abc}] \neq 0$.

5. Reciprocal system of vectors.

A. Two bases **a**, **b**, **c** and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are *reciprocal* if the nine equations (1) of art. 3·52 hold. Also then

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}.$$

Further, by symmetry $\mathbf{a} = \dfrac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']}$, etc.

Note that $[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}'] = 1$.

B. When a basis and its reciprocal are identical they are called self-reciprocal. Basis **i, j, k** is self-reciprocal and $[\mathbf{ijk}]=1$.

C. *Components.*

$$\mathbf{r}=\left.\begin{array}{l} r_1\mathbf{a}+r_2\mathbf{b}+r_3\mathbf{c} \\ r_1'\mathbf{a}'+r_2'\mathbf{b}'+r_3'\mathbf{c}' \end{array}\right\} ;$$

(r_1, r_2, r_3) are components of $\mathbf{r}$ for the basis **a, b, c** and (r_1', r_2', r_3') are the components of $\mathbf{r}$ for the reciprocal basis. Note $r_1=\mathbf{r}.\mathbf{a}'$, etc ; $r_1'=\mathbf{r}.\mathbf{a}$ etc.

$$\mathbf{q}\times\mathbf{r}=\frac{1}{[\mathbf{abc}]}\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ q_1' & q_2' & q_3' \\ r_1' & r_2' & r_3' \end{vmatrix} ; \quad \mathbf{p}.\mathbf{q}\times\mathbf{r}=\frac{1}{[\mathbf{abc}]}\begin{vmatrix} p_1' & p_2' & p_3' \\ q_1' & q_2' & q_3' \\ r_1' & r_2' & r_3' \end{vmatrix}$$

$$[\mathbf{pqr}][\mathbf{abc}]=\begin{vmatrix} \mathbf{p.a} & \mathbf{p.b} & \mathbf{p.c} \\ \mathbf{q.a} & \mathbf{q.b} & \mathbf{q.c} \\ \mathbf{r.a} & \mathbf{r.b} & \mathbf{r.c} \end{vmatrix}$$

4

Applications on Vector Products

4·1. Introduction.

In the present chapter we shall make use of the *product of vectors* in finding the equations of planes, straight lines and spheres. We shall also study the elementary properties of a tetrahedron. Finally, we shall give a few more applications in Mechanics, which involve the vector products.

4·2. Equation of a plane : Normal form.

To find the equation of a plane perpendicular to the vector **m** *and passing through the point* **a**.

Let **r** be the position vector of any point P on the plane passing through a given point **a** and perpendicular to the vector **m**. A glance at Fig. 4.1 will reveal that $\mathbf{r}-\mathbf{a}(=\overrightarrow{AP})$ will be parallel to the plane and hence perpendicular to **m**.

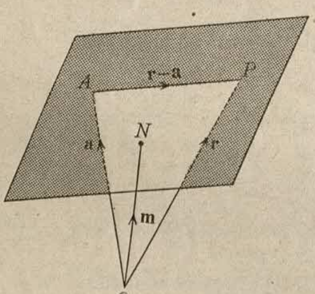


Fig. 4.1—Equation of a plane : $\mathbf{r}.\mathbf{n}=p$.

Hence

$$(\mathbf{r}-\mathbf{a}).\mathbf{m}=0 \qquad \dots \quad (1)$$

The relation (1) is true for any point P on the plane but for no point off the plane and hence it gives the equation of the required plane.

More common form :

Let **n** denote the *unit vector* perpendicular to, and *directed from the origin to the plane*. The equation of the plane will then be

$$(\mathbf{r}-\mathbf{a}).\mathbf{n}=0\ ;\ i.e.,\ \mathbf{r}.\mathbf{n}=\mathbf{a}.\mathbf{n}. \qquad \dots \quad (2)$$

But $\mathbf{a.n}$ = the length p of the perpendicular from the origin to the plane ($p = ON$ in the figure). Now (2) reduces to the common form of the equation of the plane

$$\mathbf{r.n} = p \qquad \dots \quad (3)$$

This is also known as *Normal form* of the equation of the plane.

Note 1. We again remind our readers that $\mathbf{n}$ is an unit vector and has the sense from the origin to the plane. We shall always consider p to be positive under such an assumption for the sense of $\mathbf{n}$ and thus we shall write (3) in the from $p - \mathbf{r.n} = 0$.

Important observations.

1. With reference to a system of rectangular axes through the origin O, suppose P has the coördinates (x, y, z). If the unit vector $\mathbf{n}$ has the direction cosines (l, m, n) then (3) gives

$$lx + my + nz = p,$$

the familiar form of the equation of a plane in Three-dimensional Analytic Geometry.

2. If the origin O be on the plane then (3) reduces to

$$\mathbf{r.n} = 0$$

which is, otherwise evident because the vector $\mathbf{r}$ on the plane is perpendicular to the unit normal vector $\mathbf{n}$.

3. The inclination of the two planes

$$p - \mathbf{r.n} = 0 \text{ and } p' - \mathbf{r.n'} = 0$$

is the angle θ between their normals and hence it is given by

$$\cos\theta = \mathbf{n.n'}.$$

If, instead, the equation of the planes are

$$\mathbf{r.m} = p \text{ and } \mathbf{r.m'} = p',$$

where $\mathbf{m}$ and $\mathbf{m'}$ are not unit vectors then $\cos\theta = \dfrac{\mathbf{m.m'}}{|\mathbf{m}|\ |\mathbf{m'}|}$.

We consider a plane through $P'(\mathbf{r}')$ parallel to the given plane

$$p - \mathbf{r}.\mathbf{n} = 0. \qquad \ldots \quad (1)$$

If p' be the perpendicular distance from the origin to this new plane then its equation will be

$$p' - \mathbf{r}.\mathbf{n} = 0. \qquad \ldots \quad (2).$$

Since $\mathbf{r}'$ is a point in (2) we have

$$p' - \mathbf{r}'.\mathbf{n} = 0. \qquad \ldots \quad (3)$$

Thus, as seen from Fig. 4.3, required perpendicular distance

$$= P'N = p - p' = p - \mathbf{r}'.\mathbf{n} \qquad [\text{ using } (3)\ ]$$

Sign Convention : The perpendicular distance $p - \mathbf{r}'.\mathbf{n}$ is *positive* for points on the same side of the plane as the origin but *negative* for points on the opposite side.

II. *The distance from a point* $P'(\mathbf{r}')$ *to the plane* $p - \mathbf{r}.\mathbf{n} = 0$ *measured in the direction of the unit vector* $\mathbf{b}$.

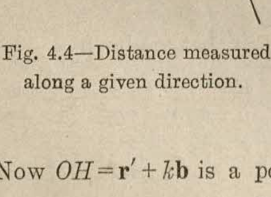


Fig. 4.4—Distance measured along a given direction.

From Fig. 4.4 it will be clear that we require the distance $P'H$ where $\overrightarrow{P'H}$ is in the direction of the unit vector $\mathbf{b}$.

Thus

$$\overrightarrow{P'H} = k\mathbf{b}$$

where $k = |\overrightarrow{P'H}|$.

Now $OH = \mathbf{r}' + k\mathbf{b}$ is a point on the plane $p - \mathbf{r}.\mathbf{n} = 0$. Hence

$$p - (\mathbf{r}' + k\mathbf{b}).\mathbf{n} = 0\ ;$$

$$\text{this gives } k = \frac{p - \mathbf{r}'.\mathbf{n}}{\mathbf{b}.\mathbf{n}}.$$

This value of k gives the required distance.

Examples. IV(B)

1. Prove that the points $\mathbf{a}_1(2, 4, -3)$ and $\mathbf{a}_2(2, -6, 2)$ are on the opposite sides of the plane $4x+7y-4z+6=0$.

2. Show that the origin and $(2, -4, 3)$ lie on different sides of the plane $x+3y-5z+7=0$.

3. Find the perpendicular distance from the point $(4, -2, 3)$ to the plane $6x+2y-9z=4$.

4. Find the distances of the points $(2, 3, 4)$ and $(1, 1, 4)$ from the plane $3x-6y+2z+11=0$.

5. (*a*) Find the incentre of the tetrahedron formed by the planes $x=0$, $y=0$, $z=0$ and $x+2y+2z=1$.

(*b*) Same problem with planes $x=0$, $y=0$, $z=0$, $x+y+z=a$.

6. Find the point where the line $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ cuts the plane $p-\mathbf{r}.\mathbf{n}=0$.

7. Show that the points $(1, -1, 3)$ and $(3, 3, 3)$ are equidistant from the plane $5x+2y-7z+9=0$ and are on the opposite sides of it.

8. Find the perpendicular distance from the point $P(1, -2, 1)$ to the plane $3x-9y+4z+25=0$.

Hints and Answers

1. First write the equation of the plane in the normal form ; thus

$$\tfrac{6}{9}+\tfrac{1}{9}(4x+7y-4z)=0.$$

Now, $p-\mathbf{a}_1.\mathbf{n}=\frac{6}{9}-(2, 4, -3).\frac{1}{9}(4, 7, -4)=6.$

Similarly $p-\mathbf{a}_2.\mathbf{n}=-4$. Hence the two points $\mathbf{a}_1$ and $\mathbf{a}_2$ are on the opposite sides of the plane.

3. Writing the equation of the plane in the normal form, we obtain

$$\tfrac{4}{11}+\tfrac{1}{11}(-6x-2y+9z)=0$$

$\therefore$ perpendicular distance from $(4, -2, 3)$ is

$$=\tfrac{4}{11}+\tfrac{1}{11}(-24+4+27)=1.$$

4. 1 ; 16/7. 5. (*a*) 1/8, 1/8, 1/8.

(*b*) $\left[\frac{a}{6}(3-\sqrt{3}),\ \frac{a}{6}(3-\sqrt{3}),\ \frac{a}{6}(3-\sqrt{3})\right]$.

6. $\mathbf{a}+\frac{p-\mathbf{a.n}}{\mathbf{b.n}}\mathbf{b}$. 8. $(25\sqrt{106})/53$.

4·22. Planes bisecting angles between two given planes.

Let the equations of two given planes be

$$p-\mathbf{r.n}=0 \qquad \cdots \quad (1)$$

$$p-\mathbf{r.n}'=0 \qquad \cdots \quad (2)$$

We are to find the equations of planes which bisect the angles between (1) and (2). We make use of the fact that any point $\mathbf{r}$ on either bisector will be equidistant from the two planes. For points on the plane bisecting the angle in which the origin lies the two perpendicular distances will have the *same sign* but for the points on the other bisector, *opposite signs*. Hence for the first bisector the equation is

$$p-\mathbf{r.n}=p'-\mathbf{r.n}'$$

i.e., $$p-p'=\mathbf{r.(n-n')} \qquad \cdots \quad (3)$$

The other bisector will have the equation

$$p+p'=\mathbf{r.(n+n')} \qquad \cdots \quad (4)$$

We observe that the angle θ between the planes (3) and (4) *i.e.*, angle between their normal vectors is given by

$$\cos\theta=\{\mathbf{(n-n').(n+n')}\}/|\mathbf{n-n'}|\,|\mathbf{n+n'}|,$$

Since $\mathbf{(n-n').(n+n')}=\mathbf{n}^2-\mathbf{n}'^2=1-1=0$, we get

$$\cos\theta=0 \text{ and consequently } \theta=90°.$$

This proves that the two bisecting planes are perpendicular to each other.

Examples. IV(C)

1. Find the equation of *that* plane bisecting the dihedral angle between the planes $4x+7y-4z+6=0$ and $2x-3y-6z=5$ *which contains the origin.*

2. Find the equations of the planes bisecting the angles between the planes $x+2y+2z-3=0$ and $3x+4y+12z+1=0$ and point out which bisects the acute angle.

3. Find the bisector of *that* angle between the planes $3x-6y+2z=-5$ and $4x-12y+3z=3$ *which contains the origin.*

Hints and Answers

1. The equation of the required bisecting plane is given by

$$\tfrac{6}{9}+\tfrac{1}{9}(4x+7y-4z)=\tfrac{5}{7}-\tfrac{1}{7}(2x-3y-6z)$$

i.e., $\quad 46x+22y-82z-3=0.$

2. The equations of the two bisecting planes are

$$\tfrac{1}{3}(x+2y+2z-3)=\pm\frac{3x+4y+12z+1}{13}$$

which gives $\quad 2x+7y-5z-21=0 \quad \cdots \text{(i)}$

$$11x+19y+31z-18=0 \quad \cdots \text{(ii)}$$

If θ be the angle between the planes (i) and the first of the given planes, we have $\cos\theta=2/\sqrt{78}$ so that $\tan\theta=\sqrt{74}/2$, which is greater than 1 and hence $\theta>45°$. Hence (i) bisects the obtuse angle and, therefore, (ii) bisects the acute angle.

3. $67x-162y+47z+44=0.$

4·23. Equations of planes satisfying different conditions : Use of Triple products.

When we require to find the equation of a plane subject to certain conditions we should proceed as :

First Step : Find a vector normal to the required plane. The given conditions will suggest the line of arguments.

Second Step : Obtain a point which passes through the required plane.

Third Step : Use form (1) of art. 4·2 to obtain the equation of the required plane.

(A) *To find the equation of a plane which is parallel to two given vectors* **c** *and* **d** *and passing through a given point* **b**.

Evidently $\mathbf{c} \times \mathbf{d}$ is a vector perpendicular to the required plane. Again the plane passes through the point **b**. Hence the required equation is

$$(\mathbf{r}-\mathbf{b}).\mathbf{c} \times \mathbf{d}=0 \; ; \; i.e., \; [\mathbf{r}-\mathbf{b}, \mathbf{c}, \mathbf{d}]=0$$

$$\text{or,} \quad [\mathbf{rcd}]=[\mathbf{bcd}].$$

Otherwise. $\mathbf{r}-\mathbf{b}$, **c**, **d** are parallel to the required plane and as such they are coplanar and hence their scalar triple product is zero. Hence etc.

(B) *To find the equation of the plane passing through three given points whose position vectors relative to an assigned origin O are* **a**, **b** *and* **c**. (Fig. 2.5, Page 53).

The vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are both parallel to the plane. Again it passes through **a**. Hence $\mathbf{r}-\mathbf{a}$ is also parallel to this plane (**r** being the position vector of any point on the plane). Similar arguments as in (A) will give the equation of the required plane as

$$(\mathbf{r}-\mathbf{a}).(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})=0 \qquad \cdots \quad (1)$$

But $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})=\mathbf{b} \times \mathbf{c}+\mathbf{c} \times \mathbf{a}+\mathbf{a} \times \mathbf{b}=\mathbf{m}$ (say) $\quad \cdots \quad (2)$

Hence (1) reduces to $(\mathbf{r}-\mathbf{a}).\mathbf{m}=0$ where **m** is given by (2).

Now $\mathbf{a}.\mathbf{m}=\mathbf{a}.(\mathbf{b} \times \mathbf{c}+\mathbf{c} \times \mathbf{a}+\mathbf{a} \times \mathbf{b})=[\mathbf{abc}]$, other terms being zero. Hence the required equation will reduce to

$$\mathbf{r}.\mathbf{m}=[\mathbf{abc}]$$

We remark here that the length of $\mathbf{m}=|\mathbf{m}|=$ twice the area of the triangle formed by the three points **a**, **b**, **c**. (*Why* ?)

(C) *To find the equation of a plane containing the line* $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ *and parallel to the vector* **c**.

The equation of the given line implies that it passes through $\mathbf{a}$ and parallel to $\mathbf{b}$. It is also parallel to $\mathbf{c}$ (given). So its equation will be

$$(\mathbf{r}-\mathbf{a}).\mathbf{b}\times\mathbf{c}=0 \ \textit{i.e.}, \ [\mathbf{rbc}]=[\mathbf{abc}].$$

(D) *To find the equation of a plane passing through the points* $\mathbf{a}$ *and* $\mathbf{b}$ *and parallal to the vector* $\mathbf{c}$.

The plane in question passes through $\mathbf{a}$ and parallel to $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}$. Hence its equation will be

$$(\mathbf{r}-\mathbf{a}).(\mathbf{b}-\mathbf{a})\times\mathbf{c}=0,$$

which, when simplified gives

$$\mathbf{r}.(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=[\mathbf{abc}].$$

(E) *To find the equation of a plane containing the line* $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ *and passing through a point* $\mathbf{c}$.

The given line passes through $\mathbf{a}$ and is parallel to $\mathbf{b}$. The required plane, therefore, passes through $\mathbf{a}$, parallel to $\mathbf{b}$ and $\mathbf{a}-\mathbf{c}$. Hence its equation will be

$$(\mathbf{r}-\mathbf{a}).(\mathbf{a}-\mathbf{c})\times\mathbf{b}=0.$$

$$\textit{i.e.,} \quad \mathbf{r}.(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c})=[\mathbf{abc}].$$

(F) *To find the equation of the plane passing through the line of intersection of the two planes* $\mathbf{r}.\mathbf{n}=p$, $\mathbf{r}.\mathbf{n}'=p'$ *and containing the point* $\mathbf{b}$.

The relation

$$\mathbf{r}.(\mathbf{n}-\lambda\mathbf{n}')=p-\lambda p' \quad \text{(where } \lambda \text{ is a parameter)}$$

represents *any* plane passing through the common points of the two given planes ; for, any point $\mathbf{r}$ satisfying the given planes will also satisfy it. Now to fix up the value of λ we are given that the required plane contains the point $\mathbf{b}$ which gives

$$\mathbf{b}.(\mathbf{n}-\lambda\mathbf{n}')=p-\lambda p' \quad \textit{i.e.,} \quad \lambda=\frac{\mathbf{b}.\mathbf{n}-p}{\mathbf{b}.\mathbf{n}'-p'}.$$

Putting this value of λ we obtain the equation of the required plane.

Examples. IV(D)

[In solving numerical problems of finding equations of planes satisfying different conditions we advise our readers not to use any of the formulæ given above. For each problem the given conditions will suggest the line of arguments. See the hints given].

1. Find the equation of the plane passing through three points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$.

2. Obtain the equation of a plane passing through three points (3, 6, 5), (4, 5, 2) and (2, 3, -1).

3. Determine the plane through the point (1, 2, -1) which is perpendicular to the line of intersection of the planes $3x - y + z = 1$ and $x + 4y - 2z = 2$.

4. Find the equation of the plane passing through the point $A(a_1, a_2, a_3)$ and parallel to the vectors (l_1, m_1, n_1) and (l_2, m_2, n_2).

5. Find the equation of the plane passing through the point (3, -2, -1) and parallel to the vectors (1, -2, 4) and (3, 2, -5).

6. Obtain the equation of the plane which contains the two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ and is parallel to the vector (l_1, m_1, n_1).

7. Find the equation of the plane containing the points (5, 2, 1) and (4, 1, -2) and parallel to the vector (3, -1, 4).

8. Determine the equation of the plane passing through three points (4, 5, 1), (0, -1, -1) and (3, 9, 4). Does the plane contain the point (-4, 4, 4) ?

9. Show that the plane containing the lines $\mathbf{r} = \mathbf{a} + t\mathbf{a}'$ and $\mathbf{r} = \mathbf{a}' + s\mathbf{a}$ is represented by $[\mathbf{r}\mathbf{a}\mathbf{a}'] = 0$. Interpret geometrically.

10. What is the equation of the plane containing the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and perpendicular to the plane $\mathbf{r}.\mathbf{c} = q$?

11. What is the equation of the plane containing two parallel lines $\mathbf{r}=\mathbf{a}+s\mathbf{b}$, $\mathbf{r}=\mathbf{a}'+t\mathbf{b}$?

12. Find the equation of the plane through $(2, -1, 6)$ and $(1, -2, 4)$ and perpendicular to the plane $x-2y-2z+9=0$.

13. Find the equation of the plane which contains the line $\mathbf{r}=t\mathbf{a}$ and is perpendicular to the plane containing the lines $\mathbf{r}=u\mathbf{b}$ and $\mathbf{r}=v\mathbf{c}$.

14. What is the equation of the plane which passes through the line of intersection of the planes $\mathbf{r}.\mathbf{n}_1=q_1$, $\mathbf{r}.\mathbf{n}_2=q_2$ and is parallel to the line of intersection of the planes $\mathbf{r}.\mathbf{n}_3=q_3$ and $\mathbf{r}.\mathbf{n}_4=q_4$?

15. Find the equation of the plane which passes through $\mathbf{a}$ and is perpendicular to the plane $\mathbf{r}=\mathbf{b}+t\mathbf{c}$.

16. Show that the four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are coplanar if and only if $[\mathbf{bcd}]+[\mathbf{cad}]+[\mathbf{abd}]=[\mathbf{abc}]$

Hints and Answers

1. Take any point $P\ (x, y, z)$ on the plane. Then

$$\overrightarrow{AP}=(x-a_1, y-a_2, z-a_3),\ \overrightarrow{AB}=(b_1-a_1, b_2-a_2, b_3-a_3),$$

$$\text{and}\quad \overrightarrow{AC}=(c_1-a_1, c_2-a_2, c_3-a_3)$$

are coplanar. Hence their scalar triple product is zero *i.e.*, $\overrightarrow{AP}\,.\,\overrightarrow{AB}\times\overrightarrow{AC}=0$ which gives the required plane in the form :

$$\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{vmatrix}=0.$$

2. $3x-9y+4z+25=0$.

3. The directions of normals to the given planes are $(3, -1, 1)$ and $(1, 4, -2)$. Their cross product is the vector $(-2, 7, 13)$; this is perpendicular to the required plane. Hence the required plane which passes through

$(1, 2, -1)$ and perp. to $(-2, 7, 13)$ is given by $(x-1, y-2, z+1).(-2, 7, 13)=0$ *i.e.*, $-2x+7y+13z+1=0$.

4. Here three vectors $(x-a_1, y-a_2, z-a_3)$, (l_1, m_1, n_1) and (l_2, m_2, n_2) are coplanar, any point on the plane being (x, y, z). The required equation will then be obtained by equating their scalar triple product to zero and hence the equation is

$$\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

5. $2x+17y+8z+36=0$.

6. $P(x, y, z)$; $\overrightarrow{AP}$, $\overrightarrow{AB}$ and the vector (l_1, m_1, n_1) are coplanar. Their scalar triple product is zero. The required equation is

$$\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0.$$

7. $7x+5y-4z=41$.

8. $5x-7y+11z+4=0$; yes.

9. The equations of the two lines give that the required plane is perpendicular to $\mathbf{a}\times\mathbf{a}'$ and also it passes through $\mathbf{a}$ and so the required equation is $(\mathbf{r}-\mathbf{a}).(\mathbf{a}\times\mathbf{a}')=0$, *i.e.*, $[\mathbf{raa}']=[\mathbf{aaa}']=0$.

10. $[\mathbf{rbc}]=[\mathbf{abc}]$.

11. $[\mathbf{r}-\mathbf{a}, \mathbf{a}-\mathbf{a}', \mathbf{b}]=0$.

12. $2x+4y-3z+26=0$.

13. $\mathbf{b}\times\mathbf{c}$ is $\parallel$ to the required plane. Also the required plane contains $\mathbf{r}=t\mathbf{a}$ which is a line passing through the origin and parallel to $\mathbf{a}$. Hence $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is normal to the required plane. Further the required plane contains the origin and hence its equation is $\mathbf{r}.\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0$.

14. Any plane passing through the line of intersection of the planes $\mathbf{r}.\mathbf{n}_1=q_1$, and $\mathbf{r}.\mathbf{n}_2=q_2$ is of the form $\mathbf{r}.(\mathbf{n}_1-\lambda\mathbf{n}_2)=q_1-\lambda q_2$ where λ is a parameter. But the required plane is parallel to $\mathbf{n}_3\times\mathbf{n}_4$ (why ?) and hence perpendicular to the normal $\mathbf{n}_1-\lambda\mathbf{n}_2$ which gives $(\mathbf{n}_1-\lambda\mathbf{n}_2).(\mathbf{n}_3\times\mathbf{n}_4)=0$.
i.e., $\lambda=[\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4]/[\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4]$. Putting the value of λ we obtain equation in the form

$$[\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4](\mathbf{r}.\mathbf{n}_1-q_1)=[\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4](\mathbf{r}.\mathbf{n}_2-q_2).$$

15. $[\mathbf{rnc}] = [\mathbf{anc}]$.

16. Four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are coplanar if and only if the point $\mathbf{d}$ lies on the plane through the three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *i.e.*, if $\mathbf{d}$ satisfies the equation $\mathbf{r}.\mathbf{m} = [\mathbf{abc}]$ (See (B) of art. 4·23) and hence the required condition is $\mathbf{d}.\mathbf{m} = [\mathbf{abc}]$ which leads to the condition required.

Otherwise. The necessary and sufficient condition for four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ to be coplanar is that there exists four scalars x, y, z, t, not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = \mathbf{0}$, $x + y + z + t = 0$.

Eliminating t, the condition becomes $x(\mathbf{a} - \mathbf{d}) + y(\mathbf{b} - \mathbf{d}) + z(\mathbf{c} - \mathbf{d}) = \mathbf{0}$. Now, x, y, z cannot all be zero, for then $t = 0$ and all four scalars are zero. Suppose $x \neq 0$, then dot-multiply by $(\mathbf{b} - \mathbf{d}) \times (\mathbf{c} - \mathbf{d})$; this gives $x(\mathbf{a} - \mathbf{d}).(\mathbf{b} - \mathbf{d}) \times (\mathbf{c} - \mathbf{d}) = 0$ and hence $(\mathbf{a} - \mathbf{d}).(\mathbf{b} - \mathbf{d}) \times (\mathbf{c} - \mathbf{d}) = 0$ $(\because x \neq 0)$. Now expand.

4·3. Equation of a straight line : Non-parametric form.

We are already familiar with the parametric form of the equation of a line *viz.* $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ which represents a line passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ and t is a parameter. We shall now give the equation of a line which will not involve any parameter.

Let us consider a line passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ and suppose any point on it be $\mathbf{r}$. Then $\mathbf{r} - \mathbf{a}$ is parallel to $\mathbf{b}$ so that

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0} \qquad \cdots \qquad (1)$$

This gives the equation of the line.

We could derive (1) from $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ by writing $\mathbf{r} - \mathbf{a} = t\mathbf{b}$ and then taking cross product with $\mathbf{b}$. Thus

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = t\mathbf{b} \times \mathbf{b} = \mathbf{0},$$

which is the same equation as (1).

More common form. We shall take $\mathbf{e}$ as the unit vector in the direction of $\mathbf{b}$. Then the equation of the line can also be written as

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{e} = \mathbf{0} \qquad \cdots \qquad (2)$$

4·31. Perpendicular distance from a point P (p) on a line $(\mathbf{r}-\mathbf{a})\times\mathbf{e}=\mathbf{0}$.

The equation of the given line implies that it passes through the point **a** and parallel to the *unit* vector **e**. If P be the point **p** (Fig. 4.5) then

$$\overrightarrow{AP}=\mathbf{p}-\mathbf{a}.$$

Suppose AP makes an angle θ with the given line. The length of the vector product of $\mathbf{p}-\mathbf{a}$ and **e** is

$$|(\mathbf{p}-\mathbf{a})\times\mathbf{e}|=AP\sin\theta=PN,$$

which is the required perpendicular distance.

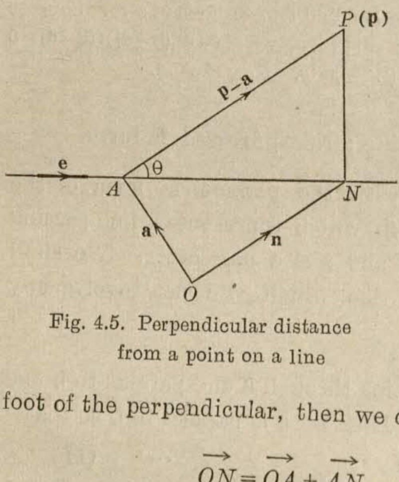


Fig. 4.5. Perpendicular distance from a point on a line

Rule. Put **p** *for* **r** *in the left member of the equation of the line given in the form* $(\mathbf{r}-\mathbf{a})\times\mathbf{e}=\mathbf{0}$.

If we require to find the position vector of N, the foot of the perpendicular, then we observe that

$$\overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}$$

i.e. $\quad \mathbf{n}=\mathbf{a}+\{(\mathbf{p}-\mathbf{a}).\mathbf{e}\}\mathbf{e}.$

Note that the length $AN=(\mathbf{p}-\mathbf{a}).\mathbf{e}$ and hence the vector

$$\overrightarrow{AN}=\{(\mathbf{p}-\mathbf{a}).\ \mathbf{e}\}\mathbf{e}$$

It is also easy to verify that

$$\overrightarrow{PN}=\overrightarrow{ON}-\overrightarrow{OP}=\mathbf{n}-\mathbf{p}=\mathbf{a}-\mathbf{p}+\{(\mathbf{p}-\mathbf{a}).\mathbf{e}\}\mathbf{e}$$

Note. It is interesting to observe that $\mathbf{r}.\mathbf{n}=0$ is the equation of a *plane* passing through the origin and *perpendicular* to **n** but $\mathbf{r}\times\mathbf{n}=\mathbf{0}$ is the equation of a *line* passing through the origin and *parallel to* **n**.

4·32. Equations of Lines satisfying different conditions :

(A) *To find the equation of a line passing through* $\mathbf{b}$ *and perpendicular to the vectors* $\mathbf{c}$ *and* $\mathbf{d}$.

The line is perpendicular to the plane containing $\mathbf{c}$ and $\mathbf{d}$ and hence parallel to $\mathbf{c} \times \mathbf{d}$. Also it passes through the point $\mathbf{b}$; hence its equation will be

$$(\mathbf{r}-\mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}.$$

(B) *To find the equation of a line passing through the point* $\mathbf{a}$ *and parallel to the line of intersection of the planes*

$$\mathbf{r}.\mathbf{n}_1 = p_1 \quad \textit{and} \quad \mathbf{r}.\mathbf{n}_2 = p_2.$$

Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the given planes, $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to their line of intersection. Thus the required line passes through $\mathbf{a}$ and parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Its equation will then be

$$(\mathbf{r}-\mathbf{a}) \times (\mathbf{n}_1 \times \mathbf{n}_2) = \mathbf{0}.$$

(C) *To find the equation of a line passing through the point* $\mathbf{a}$ *and parallel to the plane* $\mathbf{r}.\mathbf{n} = p$ *and perpendicular to the line* $\mathbf{r} = \mathbf{c} + t\mathbf{d}$.

The line is perpendicular to $\mathbf{n}$ as well as $\mathbf{d}$ and hence parallel to $\mathbf{d} \times \mathbf{n}$ and the required equation becomes

$$(\mathbf{r}-\mathbf{a}) \times (\mathbf{d} \times \mathbf{n}) = \mathbf{0}.$$

(D) *To find the equation of a straight line passing through the point* $\mathbf{c}$, *parallel to the plane* $\mathbf{r}.\mathbf{n} = p$ *and intersecting the line* $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

The required line passes through $\mathbf{c}$. Its equation will be of the form

$$(\mathbf{r}-\mathbf{c}) \times \mathbf{d} = 0 \qquad \cdots \qquad (1)$$

where $\mathbf{d}$ is a vector parallel to the required line.

To find d. The line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ intersects (1). So a plane can be drawn which will contain the

given line and the line (1). This plane will contain the points $\mathbf{a}$ and $\mathbf{c}$ and the vector $\mathbf{b}$. Hence $(\mathbf{a}-\mathbf{c})\times\mathbf{b}$ is normal to such a plane. Clearly $(\mathbf{a}-\mathbf{c})\times\mathbf{b}$ is perpendicular to the line (1).

Again the required line is parallel to the plane $\mathbf{r}.\mathbf{n}=p$ and hence it is perpendicular to the vector $\mathbf{n}$.

We now conclude that the required line (1) is parallel to $\{(\mathbf{a}-\mathbf{c})\times\mathbf{b}\}\times\mathbf{n}$. The equation (1) then reduces to

$$(\mathbf{r}-\mathbf{c})\times\{(\mathbf{a}-\mathbf{c})\times\mathbf{b}\times\mathbf{n}\}=\mathbf{0}.$$

(E) *To find the equation of the straight line passing through the point* $\mathbf{c}$, *intersecting both the lines* $\mathbf{r}=\mathbf{a}+s\mathbf{b}$ and $\mathbf{r}=\mathbf{a}'+t\mathbf{b}'$.

Since the required line intersects $\mathbf{r}=\mathbf{a}+s\mathbf{b}$ it will be perpendicular to $(\mathbf{a}-\mathbf{c})\times\mathbf{b}$ [similar arguments as in (D)]. Also it intersects the line $\mathbf{r}=\mathbf{a}'+t\mathbf{b}'$ and hence perpendicular to $(\mathbf{a}'-\mathbf{c})\times\mathbf{b}'$. The required line will then have the equation

$$(\mathbf{r}-\mathbf{c})\times\{(\mathbf{a}-\mathbf{c})\times\mathbf{b}\}\times\{(\mathbf{a}'-\mathbf{c})\times\mathbf{b}'\}=\mathbf{0}.$$

(F) *To find the equation of the line of intersection of two planes* $\mathbf{r}.\mathbf{n}_1=p_1$ *and* $\mathbf{r}.\mathbf{n}_2=p_2$.

Evidently $\mathbf{n}_1\times\mathbf{n}_2$ is parallel to the required line. In order to get *one point* through which it passes we proceed as :

Let O be the origin and $\overrightarrow{ON}$ be the perpendicular drawn from O on the line. It can easily be seen that $\overrightarrow{ON}$ lies on the plane containing $\mathbf{n}_1$ and $\mathbf{n}_2$, So $\overrightarrow{ON}$ can be expressed as a linear combination of $\mathbf{n}_1$ and $\mathbf{n}_2$, say $\overrightarrow{ON}=l\mathbf{n}_1+l'\mathbf{n}_2$, where l and l' are scalars to be determined. Again N is a common point of the two given planes. Hence

$$(l\mathbf{n}_1+l'\mathbf{n}_2).\mathbf{n}_1=p_1, \text{ and } (l\mathbf{n}_1+l'\mathbf{n}_2).\mathbf{n}_2=p_2.$$

We solve the two scalar equations for l and l' to obtain

$$l=\frac{p_1\mathbf{n}_2{}^2-p_2(\mathbf{n}_1.\mathbf{n}_2)}{\mathbf{n}_1{}^2\mathbf{n}_2{}^2-(\mathbf{n}_1.\mathbf{n}_2)^2} \text{ and } l'=\frac{p_2\mathbf{n}_1{}^2-p_1(\mathbf{n}_1.\mathbf{n}_2)}{\mathbf{n}_1{}^2\mathbf{n}_2{}^2-(\mathbf{n}_1.\mathbf{n}_2)^2}$$

Thus the required line has the equation

$\{\mathbf{r}-(l\mathbf{n}_1+l'\mathbf{n}_2)\}\times(\mathbf{n}_1\times\mathbf{n}_2)=0$ where l and l' have the values given above.

Examples. IV(E)

1. Find the equation of the line of intersection of the planes $3x-y+z=1$ and $x+4y-2z=2$.

2. Prove that the line of intersection of $x+2y+3z=0$ and $3x+2y+z=0$ is equally inclined to the axes of x and z and that it makes an angle θ with the axis of y where $\sec 2\theta=3$.

3. Find the condition that the three planes $\mathbf{r}.\mathbf{n}_1=p_1$, $\mathbf{r}.\mathbf{n}_2=p_2$, $\mathbf{r}.\mathbf{n}_3=p_3$ should have a common line of intersection.

4. Show that the planes $2x+5y+3z=0$, $x-y+4z=2$, $7y-5z+4=0$ have a common line of intersection.

5. Find the locus of a point which is equidistant from the planes $\mathbf{r}.\mathbf{n}_1=p_1$, $\mathbf{r}.\mathbf{n}_2=p_2$, $\mathbf{r}.\mathbf{n}_3=p_3$ where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are not necessarily unit vectors.

Hints and Answers

1. $(x, y, z)=\frac{1}{122}(106, 73, -23)+t(2, -7, -13)$.

2. The line is parallel to the vector $(1, 2, 3)\times(3, 2, 1)$. Now proceed.

3. The three planes are $\mathbf{r}.\mathbf{n}_1=p_1$ ... (1), $\mathbf{r}.\mathbf{n}_2=p_2$ ... (2), $\mathbf{r}.\mathbf{n}_3=p_3$ ... (3). The equation of a plane passing through the line of intersection of (1) and (2) may be written as

$$\mathbf{r}.(\mathbf{n}_1-\lambda\mathbf{n}_2)=p_1-\lambda p_2 \qquad \ldots (4)$$

where λ is a parameter.

If the three planes (1), (2), (3) have a common line of intersection then there exists a value for λ for which (4) would be identical with (3). That is, $\mathbf{n}_1-\lambda\mathbf{n}_2=k\mathbf{n}_3$ ... (5) and $p_1-\lambda p_2=kp_3$ ... (6), where k is a scalar constant. In order to solve for λ and k we proceed as :

From (5), $$(\mathbf{n}_1-\lambda\mathbf{n}_2)\times\mathbf{n}_3=k\mathbf{n}_3\times\mathbf{n}_3=0$$

i.e., $$\lambda\mathbf{n}_2\times\mathbf{n}_3=\mathbf{n}_1\times\mathbf{n}_3 \qquad \ldots (7)$$

Also from (5), $$(\mathbf{n}_1-\lambda\mathbf{n}_2)\times\mathbf{n}_2=k\mathbf{n}_3\times\mathbf{n}_2$$

i.e., $$k\mathbf{n}_3\times\mathbf{n}_2=\mathbf{n}_1\times\mathbf{n}_2 \qquad \ldots (8)$$

Thus from (6), $$(p_1-\lambda p_2)(\mathbf{n}_2\times\mathbf{n}_3)=kp_3\,(\mathbf{n}_2\times\mathbf{n}_3)$$

i.e., $$p_1(\mathbf{n}_2\times\mathbf{n}_3)-\lambda p_2(\mathbf{n}_2\times\mathbf{n}_3)=kp_3(\mathbf{n}_2\times\mathbf{n}_3)$$

i.e., $$p_1(\mathbf{n}_2\times\mathbf{n}_3)-p_2(\mathbf{n}_1\times\mathbf{n}_3)=-p_3(\mathbf{n}_1\times\mathbf{n}_2), \text{ using (7) and (8).}$$

i.e., $$p_1(\mathbf{n}_2\times\mathbf{n}_3)+p_2(\mathbf{n}_3\times\mathbf{n}_1)+p_3(\mathbf{n}_1\times\mathbf{n}_2)=0.$$

This is the required condition.

4. Proceed as in Ex. 3 above and show that the condition is satisfied. *Otherwise.* Observe that :

$$\text{equation (1)}-2\text{ equation (2)}=\text{equation (3)}.$$

Hence etc.

5. Locus is a straight line determined by the two planes

$$\mathbf{r}.\left(\frac{\mathbf{n}_1}{|\mathbf{n}_1|}-\frac{\mathbf{n}_2}{|\mathbf{n}_2|}\right)=\frac{p_1}{|\mathbf{n}_1|}-\frac{p_2}{|\mathbf{n}_2|};$$

and $$\mathbf{r}.\left(\frac{\mathbf{n}_2}{|\mathbf{n}_2|}-\frac{\mathbf{n}_3}{|\mathbf{n}_3|}\right)=\frac{p_2}{|\mathbf{n}_2|}-\frac{p_3}{|\mathbf{n}_3|}.$$

4·33. Condition of intersection of two lines $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, $\mathbf{r}=\mathbf{c}+s\mathbf{d}$.

The given lines contain the points **a** and **c** and are parallel to **b** and **d** respectively. If they intersect, their common plane will be parallel to each of the vectors **b**, **d** and $\mathbf{a}-\mathbf{c}$ and as such their scalar triple product will vanish. *i.e.*,

$$[\mathbf{b},\mathbf{d},\mathbf{a}-\mathbf{c}]=0$$

which gives the required condition.

4·34. Two useful Resolutions and their applications.

(A) *If* **a**, **b**, **c** *be any three non-coplanar vectors then* $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$ *are also non-coplanar. Further it is required to express* **a**, **b**, **c** *in terms of* $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ *and* $\mathbf{a}\times\mathbf{b}$.

Solution. Since **a**, **b**, **c** are non-coplanar,

$$[\mathbf{abc}]\neq 0 \qquad \cdots \quad (1)$$

Now we know that

$$[\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}]=[\mathbf{abc}]^2.$$

Hence the left-hand box does not vanish by (1) and consequently $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$ are three non-coplanar vectors.

Second Part. Any vector **a** can be expressed as a linear combination of three non-coplanar vectors $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$. Suppose

$$\mathbf{a}=l\mathbf{b}\times\mathbf{c}+m\mathbf{c}\times\mathbf{a}+n\mathbf{a}\times\mathbf{b} \qquad \cdots \quad (2)$$

where l, m, n are scalars to be found out. Hence

$$\mathbf{a}.\mathbf{a}=l\mathbf{a}.\mathbf{b}\times\mathbf{c}+m\mathbf{a}.\mathbf{c}\times\mathbf{a}+n\mathbf{a}.\mathbf{a}\times\mathbf{b}$$
$$=l[\mathbf{abc}],\ \text{other terms will vanish.}$$

Thus $l=\dfrac{\mathbf{a}.\mathbf{a}}{[\mathbf{abc}]}$. Similarly $m=\dfrac{\mathbf{a}.\mathbf{b}}{[\mathbf{abc}]}$, $n=\dfrac{\mathbf{a}.\mathbf{c}}{[\mathbf{abc}]}$

Putting the values of l, m, n in (2) we get

$$\mathbf{a}=\frac{\mathbf{a}.\mathbf{a}}{[\mathbf{abc}]}\mathbf{b}\times\mathbf{c}+\frac{\mathbf{a}.\mathbf{b}}{[\mathbf{abc}]}\mathbf{c}\times\mathbf{a}+\frac{\mathbf{a}.\mathbf{c}}{[\mathbf{abc}]}\mathbf{a}\times\mathbf{b}.$$

Similarly we can express **b** and **c** in terms of

$$\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}\ \text{and}\ \mathbf{a}\times\mathbf{b}.$$

(B) *If* **a**, **b** *and* **c** *be three non-coplanar vectors then it is required to express the three non-coplanar vectors* $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ *and* $\mathbf{a}\times\mathbf{b}$ *in terms of* **a**, **b** *and* **c**.

Solution. Since **a, b, c** are three non-coplanar vectors we may write,

$$\mathbf{b}\times\mathbf{c}=l\mathbf{a}+m\mathbf{b}+n\mathbf{c}.$$

Dot-multiply both sides by $\mathbf{b} \times \mathbf{c}$; then

$$\mathbf{b} \times \mathbf{c}.\mathbf{b} \times \mathbf{c} = l\mathbf{a}.\mathbf{b} \times \mathbf{c}.$$

This gives $l = \dfrac{(\mathbf{b} \times \mathbf{c}).(\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]}$.

Similarly $m = \dfrac{(\mathbf{b} \times \mathbf{c}).(\mathbf{c} \times \mathbf{a})}{[\mathbf{abc}]}$, $n = \dfrac{(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{b})}{[\mathbf{abc}]}$

Therefore, we obtain

$$\mathbf{b} \times \mathbf{c} = \frac{(\mathbf{b} \times \mathbf{c}).(\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]}\mathbf{a} + \frac{(\mathbf{b} \times \mathbf{c}).(\mathbf{c} \times \mathbf{a})}{[\mathbf{abc}]}\mathbf{b} + \frac{(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{b})}{[\mathbf{abc}]}\mathbf{c}.$$

Similarly we can express $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Applications.

I. *Find the condition that the lines*

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \times \mathbf{c}, \quad \mathbf{r} = \mathbf{b} + p(\mathbf{c} \times \mathbf{a})$$

may intersect. Accepting that condition, find the point of intersection; $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are any three non-coplanar vectors.*

Solution. Using art. 4·33, the required condition becomes

$$[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} - \mathbf{b}] = 0.$$

On simplification, the required condition beomes $\mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c}$.

To find the point of intersection.

Let us write $\mathbf{a}$ and $\mathbf{b}$ in terms of $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$, the expressions are given by (A) above. Rewrite the equations of two given lines as

$$\mathbf{r} = \frac{1}{[\mathbf{abc}]}\{(\mathbf{a}.\mathbf{a})\mathbf{b} \times \mathbf{c} + (\mathbf{a}.\mathbf{b})\mathbf{c} \times \mathbf{a} + (\mathbf{a}.\mathbf{c})\mathbf{a} \times \mathbf{b}\} + t\mathbf{b} \times \mathbf{c}$$

and $\mathbf{r} = \dfrac{1}{[\mathbf{abc}]}\{(\mathbf{b}.\mathbf{a})\mathbf{b} \times \mathbf{c} + (\mathbf{b}.\mathbf{b})\mathbf{c} \times \mathbf{a} + (\mathbf{b}.\mathbf{c})\mathbf{a} \times \mathbf{b}\} + p\mathbf{c} \times \mathbf{a}$.

When they intersect they have a common value for $\mathbf{r}$. Now equating the coefficient of $\mathbf{b} \times \mathbf{c}$ we get

$$\frac{\mathbf{a}.\mathbf{a}}{[\mathbf{abc}]} + t = \frac{\mathbf{b}.\mathbf{a}}{[\mathbf{abc}]}.$$

Putting this value of t, we get the point of intersection as

$$\mathbf{r}=\frac{1}{[\mathbf{abc}]}\{(\mathbf{b.a})\mathbf{b}\times\mathbf{c}+(\mathbf{a.b})\mathbf{c}\times\mathbf{a}+(\mathbf{a.c})\mathbf{a}\times\mathbf{b}\}.$$

II. *Find the point of intersection of the three planes*

$$\mathbf{r.n}_1=p_1,\ \mathbf{r.n}_2=p_2 \text{ and } \mathbf{r.n}_3=p_3.$$

where $\mathbf{n}_1$, $\mathbf{n}_2$ *and* $\mathbf{n}_3$ *are three non-coplanar vectors, not necessarily unit vectors.*

Solution. Let us express the point of intersection $\mathbf{r}$ in terms of three non-coplanar vectors $\mathbf{n}_2\times\mathbf{n}_3$, $\mathbf{n}_3\times\mathbf{n}_1$, $\mathbf{n}_1\times\mathbf{n}_2$ as

$$\mathbf{r}=u(\mathbf{n}_2\times\mathbf{n}_3)+v(\mathbf{n}_3\times\mathbf{n}_1)+w(\mathbf{n}_1\times\mathbf{n}_2),$$

where u, v and w are scalars. Substituting in $\mathbf{r.n}_1=p_1$, we obtain

$$u\mathbf{n}_2\times\mathbf{n}_3.\mathbf{n}_1=p_1 \text{ and so } u=\frac{p_1}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}.$$

Substituting in other equations, we similarly obtain

$$v=\frac{p_2}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]},\ w=\frac{p_3}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}.$$

The required point of intersection will be given by

$$\mathbf{r}=\frac{1}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}(p_1\mathbf{n}_2\times\mathbf{n}_3+p_2\mathbf{n}_3\times\mathbf{n}_1+p_3\mathbf{n}_1\times\mathbf{n}_2).$$

4·35. Shortest distance (S.D.) between two skew lines.

Let the equations of two straight lines be

$$\mathbf{r}=\mathbf{a}+t\mathbf{b},\ \mathbf{r}=\mathbf{c}+s\mathbf{d}.$$

These two lines are respectively parallel to the vectors $\mathbf{b}$ and $\mathbf{d}$. Then $\mathbf{b}\times\mathbf{d}$ is perpendicular to both and hence parallel to their common perpendicular PN (Fig. 4.6). The given equations also imply that the points A (position vector $\mathbf{a}$)

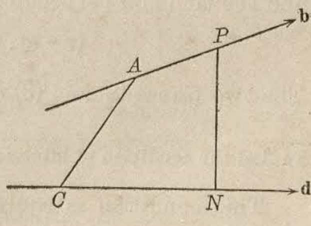


Fig. 4.6. Shortest distance between two skew lines

and C (position vector $\mathbf{c}$) lie on them, one on each line. The *shortest distance* (S. D.) p between the lines is then the length of projection of CA on $\mathbf{n}$, the unit vector in the direction of $\mathbf{b} \times \mathbf{d}$.

Hence

$$p = \overrightarrow{CA} \, . \, \mathbf{n} = (\mathbf{a} - \mathbf{c}) \cdot \frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|} \qquad \ldots \quad (1)$$

If, again, two non-parallel lines are given by the points A, B and C, D then we can find the shortest distance between them by first finding the vector $\overrightarrow{AB} \times \overrightarrow{CD}$ which is perpendicular to both the lines and then finding the unit vector $\mathbf{n}$ in this direction. Next we take any vector $\mathbf{u}$ from one line to the other (as $\overrightarrow{AC}$), p is then numercially equal to $\mathbf{n}.\mathbf{u}$. We may compute $\mathbf{n}.\overrightarrow{BD}$ and check the result.

To find the equation of the line PN.

The line PN is the line of intersection of the two planes drawn through the given lines and the line PN. The equation of the plane containing the first line and PN is

$$(\mathbf{r} - \mathbf{a}) \, . \, \mathbf{b} \times (\mathbf{b} \times \mathbf{d}) = 0 \qquad \ldots \quad (2)$$

The point N is that in which the second line meets this plane. Similarly the equation of the plane containing the second line and the common perpendicular to the two lines is

$$(\mathbf{r} - \mathbf{c}) \, . \, \mathbf{d} \times (\mathbf{b} \times \mathbf{d}) = 0 \qquad \ldots \quad (3)$$

The two planes (2) and (3) determine the line PN.

To find the condition of intersection of two lines.

This condition is equivalent to the vanishing of p. Hence from (1) the condition becomes (compare art. 4·33).

$$(\mathbf{a} - \mathbf{c}) \, . \, \mathbf{b} \times \mathbf{d} = 0$$

$$\text{or,} \quad [\mathbf{b}, \mathbf{d}, \mathbf{a} - \mathbf{c}] = 0.$$

Examples. IV(F)

1. Find the S.D. between the two lines through $A(6, 2, 2)$ and $C(-4, 0, -1)$ and parallel to the vectors $(1, -2, 2)$ and $(3, -2, -2)$ respectively. Find where the lines meet the common perpendicular.

2. (*a*) Find the S.D. between the two lines, one joining the points $A(-1, 2, -3)$ and $B(-16, 6, 4)$ and the other joining the points $C(1, -1, 3)$ and $D(4, 9, 7)$.

(*b*) Examine similarly the two lines given by the points $A(1, -2, -1)$, $B(4, 0, -3)$ and $C(1, 2, -1)$, $D(2, -4, -5)$.

3. Find the S. D. between the lines

$$\tfrac{1}{2}(x-1)=\tfrac{1}{3}(y-2)=\tfrac{1}{4}(z-3)\ ;\ \tfrac{1}{3}(x-2)=\tfrac{1}{4}(y-3)=\tfrac{1}{5}(z-4).$$

Are the lines coplanar ?

4. Find where the S. D. between the lines

$$\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}\ ;\ \frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$$

meets them.

5. Find the S. D. between the lines determined by the equations

$$3x-4y-z+5=0=3x-6y-2z+13$$
$$3x+4y-3z+2=0=3x-2y+6z+17$$

and also find where the lines meet the common perpendicular between them.

6. The S. D. between the diagonals of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it are

$$\frac{bc}{\sqrt{b^2+c^2}},\ \frac{ca}{\sqrt{c^2+a^2}},\ \frac{ab}{\sqrt{a^2+b^2}}.$$

7. Find the S. D. between the lines $\mathbf{r}=t\mathbf{k}$, $\mathbf{r}=\mathbf{a}+s\mathbf{b}$ and determine the equation of the line which cuts both the lines at right angles.

8. A straight line intersects two non-coplanar straight lines, and moves parallel to a fixed plane. Prove that the locus of a point which divides the intercept in a constant ratio is a straight line.

9. The locus of the middle points of all straight lines terminated by two fixed non-intersecting straight lines is a plane bisecting their common perpendicular at right angles.

10. In each of the three planes given by two of the lines OA, OB, OC a straight line is drawn through O perpendicular to the third line ; prove that these three lines are coplanar.

11. Two skew lines AP and BQ are met by the line of S. D. between them at A and B respectively and P, Q are points on them such that $AP=p$, $BQ=q$. If the planes APQ and BPQ are perpendicular, prove that pq is constant.

Hints and Answers

1. Here $\mathbf{a}=(6, 2, 2)$ and $\mathbf{c}=(-4, 0, -1)$ and so $\mathbf{a}-\mathbf{c}=(10, 2, 3)$. Again $\mathbf{b}=(1, -2, 2)$ and $\mathbf{d}=(3, -2, -2)$. Hence $\mathbf{b}\times\mathbf{d}=(8, 8, 4)$; $\mathbf{n}=\dfrac{\mathbf{b}\times\mathbf{d}}{|\mathbf{b}\times\mathbf{d}|}=\dfrac{1}{12}(8, 8, 4)$ $=\frac{1}{3}(2, 2, 1)$. Thus $p=(\mathbf{a}-\mathbf{c})\cdot\mathbf{n}=\frac{1}{3}(20+4+3)=9$.

The plane APN is given by

$$(x-6, y-2, z-2)\cdot(1, -2, 2)\times(8, 8, 4)=0$$

i.e., $2x-y-2z=6$.

The line CN has the equation

$$(x, y, z)=(-4, 0, -1)+t(3, -2, -2)$$

i.e., $\dfrac{x+4}{3}=\dfrac{y}{-2}=\dfrac{z+1}{-2}=t$

This line meets the plane APN at the point N whose coördinates can be easily obtained as $(-1, -2, -3)$. Similarly obtain the point $P=(5, 4, 0)$.

2. (*a*) S. D. $=7$. (*b*) S. D. $=4/3$.

3. S. D. $=0$ and thus coplanar.

4. $(11, 11, 31)$ and $(3, 5, 7)$.

5. The first line, being the intersection of two planes, is perpendicular to both normals and therefore has the direction of the vector $\mathbf{b}=(2, 3, -6)$. *One* point on the line is $\mathbf{a}\,(11/3, 4, 0)$ (point of intersection of the line with the plane $z=0$). Similarly the second line is parallel to the vector $\mathbf{d}=(2, -3, -2)$ and *one* point on the line is $\mathbf{c}(-4, 5/2, 0)$. Then $\mathbf{b}\times\mathbf{d} = -4(6, 2, 3)$ and $\mathbf{n}=-\frac{1}{7}(6, 2, 3)$; $\mathbf{a}-\mathbf{c}=(23/3, 3/2, 0)$, and so

$$p=-\tfrac{1}{7}(6, 2, 3).(23/3, 3/2, 0)=-7.$$

Proceed as in Ex. 1 and obtain $P(3, 3, 2)$ and $N(-3, 1, -1)$.

6. Draw a rectangular parallelopiped as in Fig. 1.18 but use the letters as in Fig. 1.17. Suppose we require to find the S. D. between OP and AP'. Their equations are respectively $\mathbf{r}=t(\mathbf{a}+\mathbf{b}+\mathbf{c})$ and $\mathbf{r}=\mathbf{a}+l\mathbf{b}$. Hence the S. D. between them is given by

$$\frac{\{(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}\}.\{\mathbf{o}-\mathbf{a}\}}{|(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}|}=-\frac{\mathbf{c}\times\mathbf{b}.\mathbf{a}}{|(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}|}=-\frac{abc}{b\sqrt{c^2+a^2}}.$$

$\therefore$ numerical value of the S. D. $=\dfrac{ca}{\sqrt{c^2+a^2}}$. Similarly obtain others.

7. S. D. $=[\mathbf{abk}]/|\mathbf{k}\times\mathbf{b}|$. The required line is the line of intersection of the planes $[\mathbf{r}, \mathbf{k}, \mathbf{k}\times\mathbf{b}]=0$ and $[\mathbf{r}-\mathbf{a}, \mathbf{b}, \mathbf{k}\times\mathbf{b}]=0$.

8. Take O, the mid-point of the line of S. D. PN as origin (compare Fig. 4.6). Suppose $\overrightarrow{OP}=\mathbf{a}$ then $\overrightarrow{ON}=-\mathbf{a}$. Let $\mathbf{b}$ and $\mathbf{d}$ be two vectors parallel to the given lines (as in art. 4·35). The equations of the lines will be $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ and $\mathbf{r}=-\mathbf{a}+l\mathbf{d}$. Suppose the fixed plane has the equation $\mathbf{r}.\mathbf{n}=p$. Two points A and C, one on each line has the position vectors $\mathbf{a}+t\mathbf{b}$, $-\mathbf{a}+l\mathbf{d}$ (of course t, l are fixed constants). The join of A and C will be parallel to the fixed plane if

$$\{\mathbf{a}+t\mathbf{b}-(-\mathbf{a}+l\mathbf{d})\}\,.\,\mathbf{n}=0 \qquad \cdots \quad (1)$$

Take a point D on AC such that $AD : DC=x : y$. The position vector of D is given by

$$\mathbf{r}=\frac{1}{x+y}\{y(\mathbf{a}+t\mathbf{b})+x(-\mathbf{a}+l\mathbf{d})\} \qquad \cdots \quad (2)$$

Equation (1) will give l in terms of t. Put that value of l in (2) and then obtain (2) in the form $\mathbf{r}=\mathbf{c}+\lambda\mathbf{e}$ where $\mathbf{c}$ and $\mathbf{e}$ are vectors involving x, y, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$. This will imply that the locus is a straight iine.

10. Take O as origin. Suppose $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$. The plane containing OA and OB is $\mathbf{r}\,.\,\mathbf{a}\times\mathbf{b}=0$. The line which lies in this plane and is perpendicular to OC is parallel to $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. Other lines can be similarly

shown to be parallel to $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$ and $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$. Now use the identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, so that three vectors $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, etc. are coplanar. Hence the problem.

4·4. Volume of a tetrahedron.

Consider the tetrahedron $ABCD$ (Fig. 4.7) and let the triangle ABC be the base and D be its fourth vertex. Suppose DN is drawn perpendicular to the plane face ABC so that DN is the altitude of the tetrahedron.

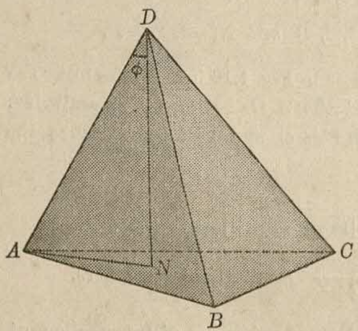


Fig. 4.7
Volume of a tetrahedron

The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane ABC and is parallel to ND.

$$\text{Its length} = |\overrightarrow{AB} \times \overrightarrow{AC}|$$
$$= AB \, . \, AC \sin BAC$$
$$= 2\triangle ABC$$

$$\therefore \text{ Volume of the tetrahedron} = \tfrac{1}{3} \text{ area of } \triangle ABC \times \text{height } DN$$
$$= \tfrac{1}{3}\{\tfrac{1}{2}AB.AC \sin BAC\}\, AD \cos \phi$$
$$= \tfrac{1}{6}\, \overrightarrow{AB} \times \overrightarrow{AC}.\overrightarrow{AD}$$

We shall usually consider the numerical value of the above expression as the volume of the tetrahedron.

Particular cases.

I. Take A as origin and suppose $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$, $\overrightarrow{AD} = \mathbf{d}$. The volume of the tetrahedron will then be the numerical value of

$$\tfrac{1}{6}\, \mathbf{b} \times \mathbf{c}.\mathbf{d} = \tfrac{1}{6}[\mathbf{bcd}].$$

II. Take any arbitrary point O as origin and let the vertices have the position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ with reference to O. Then the volume of the tetrahedron will be the numerical value of

$$\tfrac{1}{6}[\mathbf{b} - \mathbf{a},\ \mathbf{c} - \mathbf{a},\ \mathbf{d} - \mathbf{a}]$$
$$= \tfrac{1}{6}[\mathbf{bcd}] + [\mathbf{cad}] + [\mathbf{dab}] - [\mathbf{abc}].$$

III. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the coördinates of the vertices of the tetrahedron $ABCD$ with reference to a system of rectangular axes through O. If, as usual, we denote the unit vectors along the axes by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then evidently,

$$\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \text{ etc. and}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}, \text{ etc.}$$

$\therefore$ Volume V of the tetrahedron

$$= \tfrac{1}{6}\overrightarrow{AB} \times \overrightarrow{AC} \,.\, \overrightarrow{AD} = \tfrac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}$$

It is easy to verify by the properties of determinants that

$$V = \tfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

If, however, the axes through O are not mutually perpendicular and if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the unit vectors along the three axes then we have

$$\overrightarrow{OA} = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}, \text{ etc. and}$$

$$\overrightarrow{AB} = (x_2 - x_1)\,\mathbf{a} + (y_2 - y_1)\,\mathbf{b} + (z_2 - z_1)\,\mathbf{c}, \text{ etc.}$$

so that the required volume V is the numerical value of

$$\tfrac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} [\mathbf{abc}]$$

Corollary. The S.D. between two opposite edges AC and BD of the tetrahedron $ABCD$ is the numerical value of

$$\frac{(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{b}).(\mathbf{d}-\mathbf{a})}{|(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{b})|}$$

For, the edges AC, BD are respectively parallel to $\mathbf{c}-\mathbf{a}$, $\mathbf{d}-\mathbf{b}$ and the points $\mathbf{a}$, $\mathbf{d}$ lie one on each line ; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are supposed to be the position vectors of the four edges.

Examples. IV(G)

1. Find the volume of the tetrahedron whose vertices are

$$A(2,\ -1,\ -3),\ B(4,\ 1,\ 3),\ C(3,\ 2,\ -1) \text{ and } D(1,\ 4,\ 2).$$

2. A tetrahedron has the vertices

$$A(0,\ 0,\ 0),\ B(1,\ 1,\ 0),\ C(0,\ 1,\ -1) \text{ and } D(1,\ 0,\ -1).$$

Prove that the vector $(1, 1, -1)$ is perpendicular to the face BCD. Also show that the three lines *viz.* AB, perpendicular from A on BCD and perpendicular from B on ACD, are coplanar.

3. Show that the volume of the tetrahedron bounded by the four planes $my+nz=0$, $nz+lx=0$, $lx+my=0$ and $lx+my+nz=p$ is the absolute value of $2p^3/3lmn$.

4. $ABCD$ is a tetrahedron ; p_1, p_2, p_3, p_4 are perpendicular distances of the vertices from the opposite faces and s_1, s_2, s_3 are the S.D. between the three pairs of opposite edges, prove that

$$\Sigma 1/s_1{}^2 = \Sigma 1/p_1{}^2.$$

5. In a tetrahedron if two pairs of opposite edges are perpendicular, the third pair is also perpendicular to each other and the sum of the squares on the opposite edges is the same for each pair.

6. Show that in the tetrahedron of the previous example the straight line joining any vertex to the orthocentre of the

opposite face is perpendicular to that face and these four perpendiculars are concurrent.

7. Show that the sum of the squares of the edges of any tetrahedron is equal to four times the sum of the squares on the joins of the midpoints of the opposite edges.

8. Find the volume of a tetrahedron in terms of lengths of three edges which meet in a point and the angles which they make with each other.

9. Find the volume of the tetrahedron formed by planes whose equations are $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=1$.

10. Prove that the volume of the tetrahedron $OABC$ (Fig. 3.11) formed by the centroids of the faces are in the ratio of 27 : 1.

11. If four lines joining the corresponding vertices of two tetrahedron are concurrent, the lines of intersection of the corresponding faces are coplanar and *conversely*.

12. Lines are drawn in a given direction through the vertices O, A, B, C of a tetrahedron $OABC$ to meet the opposite face in O', A', B', C'. Prove that

$$\frac{\text{volume of } O'A'B'C'}{\text{volume of } OABC}=3.$$

13. Show that the six planes, each passing through one edge of a tetrahedron and bisecting the opposite edge, meet in a point.

14. Prove that the six planes bisecting the angles between consecutive faces of a tetrahedron meet in a point.

Hints and Answers

1. Volume $=7\frac{1}{3}$. 2. See hints given for No. 9.

5. Take A as origin. See Fig. 4.7. The position vectors of B, C, D with

reference to A are $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ (say). Then $\overrightarrow{BD}=\mathbf{d}-\mathbf{b}$, $\overrightarrow{AC}=\mathbf{c}$. Suppose BD is perpendicular to AC and AD is perpendicular to BC. Then

$$(\mathbf{d}-\mathbf{b}).\mathbf{c}=0, \quad i.e., \quad \mathbf{d}.\mathbf{c}=\mathbf{b}.\mathbf{c}$$

$$\text{and} \quad (\mathbf{c}-\mathbf{b}).\mathbf{d}=0, \quad i.e., \quad \mathbf{d}.\mathbf{c}=\mathbf{b}.\mathbf{d}$$

$$\text{Hence} \quad \mathbf{d}.\mathbf{c}=\mathbf{b}.\mathbf{c}=\mathbf{b}.\mathbf{d} \quad \cdots (1);$$

this gives $\mathbf{b}.(\mathbf{c}-\mathbf{d})=0$ *i.e.*, AB is perpendicular to CD.

Second part. The sum of the squares on BD and AC is $\mathbf{d}^2+\mathbf{b}^2-2\mathbf{b}.\mathbf{d}+\mathbf{c}^2$. In view of (1) the result follows.

8. We refer to Fig. 3.11, where $OABC$ is the tetrahedron with the three edges OA, OB, OC concurrent at O. Taking O as origin, let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$; $\angle BOC=\lambda$, $\angle COA=\mu$, $\angle AOB=\nu$.

$$\text{Volume } V \text{ of the tetrahedron} = \tfrac{1}{6}\,|\,[\mathbf{abc}]\,|\,.$$

$$\text{Now } [\mathbf{abc}]^2=\begin{vmatrix} \mathbf{a}.\mathbf{a} & \mathbf{a}.\mathbf{b} & \mathbf{a}.\mathbf{c} \\ \mathbf{b}.\mathbf{a} & \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{c} \\ \mathbf{c}.\mathbf{a} & \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{c} \end{vmatrix} \quad (\text{art. } 3{\cdot}5.\ \text{Illus. } 9)$$

$$=\begin{vmatrix} a^2 & ab\cos\nu & ac\cos\mu \\ ab\cos\nu & b^2 & bc\cos\lambda \\ ac\cos\mu & bc\cos\lambda & c^2 \end{vmatrix}$$

$$=a^2b^2c^2\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

$$\therefore \text{ required volume } V=\tfrac{1}{6}\,abc\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{\frac{1}{2}},$$

where the positive square root should be considered.

9. Obtain the point of intersection of first three planes *viz.* (0, 0, 0) so that (0, 0, 0) is one of the vertices of the tetrahedron. Similarly obtain the other three vertices *viz.*, $(-1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$. Now use the determinant of art. 4·4, case III. The required volume $=2/3$.

13. See Fig 4.7. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be the position vectors of the vertices with reference to some origin O. The plane containing AB and bisecting DC

contains the points $\mathbf{a}, \mathbf{b}, \frac{1}{2}(\mathbf{c}+\mathbf{d})$. If $\mathbf{r}$ be the position vector of any point on this plane then the vectors $\mathbf{r}-\mathbf{a}$, $\mathbf{b}-\mathbf{a}$, $\frac{1}{2}(\mathbf{c}+\mathbf{d})-\mathbf{a}$ are coplanar and hence

$$[\mathbf{r}-\mathbf{a}, \mathbf{b}-\mathbf{a}, \tfrac{1}{2}(\mathbf{c}+\mathbf{d})-\mathbf{a}]=0 \qquad \cdots \quad (1)$$

This gives the equation of the plane. We next verify that a point $\mathbf{p}$ given by $\mathbf{p}=\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$ lies on the plane (1). It follows that the six planes, each passing through one edge and bisecting the opposite edge meet at $\mathbf{p}$ (by symmetry).

4·41. Properties of a regular tetrahedron.

DEFINITION. A tetrahedron whose edges are all equal is called a *regular tetrahedron.* The faces of a regular tetrahedron are equilateral triangles and hence the angle between any two of its concurrent edges is $60°$.

Properties.

I. The angle θ between any two plane faces is given by $\cos\theta=\frac{1}{3}$.

II. The angle θ between any *edge* and the *face not containing that* edge is given by $\cos\theta=1/\sqrt{3}$.

III. Any two opposite edges are perpendicular to each other. The S.D. between the two opposite edges is equal to half the diagonal of the square described on an edge.

IV. The distance of any vertex from the opposite face is $\sqrt{\frac{2}{3}}\,a$, a being the length of any edge.

V. The perpendiculars from the vertices to the opposite faces meet the faces at their centroids.

Proofs. Consider a regular tetrahedron $DABC$ (Fig. 4.7) and take A as origin. With reference to A, suppose the position vectors of B, C, D be $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively. Now since the tetrahedron is regular, we have

$$\left.\begin{aligned} &|\mathbf{b}|=|\mathbf{c}|=|\mathbf{d}|=a \text{ (say)};\\ &|\mathbf{b.c}|=|\mathbf{c.d}|=|\mathbf{d.b}|=a^2\cos 60°=\tfrac{1}{2}a^2;\\ &\mathbf{b.b}=\mathbf{c.c}=\mathbf{d.d}=a^2 \end{aligned}\right\} \qquad (A)$$

I. Suppose we require to find the angle θ between two plane faces ABC and ABD. Evidently $\mathbf{b} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{d}$ are in the directions of normals to the two planes. Since the angle between two planes is the angle between their normals, we have

$$\cos \theta = \frac{(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{c}| \, |\mathbf{b} \times \mathbf{d}|}$$

But $$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) = \begin{vmatrix} \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{d} \\ \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{d} \end{vmatrix} = \begin{vmatrix} a^2 & \frac{1}{2}a^2 \\ \frac{1}{2}a^2 & \frac{1}{2}a^2 \end{vmatrix} = \tfrac{1}{4}a^4.$$

Again $|\mathbf{b} \times \mathbf{c}| = |\mathbf{b}| \, |\mathbf{c}| \sin 60^\circ = \frac{1}{2}\sqrt{3}a^2 = |\mathbf{b} \times \mathbf{d}|$ so that

$$\cos \theta = \tfrac{1}{4}a^4 / (\tfrac{1}{2}\sqrt{3}a^2)^2 = \tfrac{1}{3}.$$

Similarly we can find the angle between any other pair of plane faces.

II. Suppose now we require the angle θ between any plane face ABC and the edge BD. The direction of normal to the plane ABC is given by $\mathbf{b} \times \mathbf{c}$. The equation of the line BD is

$$\mathbf{r} = \mathbf{b} + t(\mathbf{d} - \mathbf{b}).$$

The angle between a line and a plane is the *complement* of the angle between the line and normal to the plane. Hence,

$$\cos\left(\tfrac{1}{2}\pi - \theta\right) = \frac{(\mathbf{b} \times \mathbf{c}).(\mathbf{d} - \mathbf{b})}{|\mathbf{b} \times \mathbf{c}| \; |\mathbf{d} - \mathbf{b}|} = \frac{[\mathbf{bcd}]}{|\mathbf{b} \times \mathbf{c}| \; |\mathbf{d} - \mathbf{b}|} \quad \cdots \quad (1)$$

But, we know that $$[\mathbf{bcd}]^2 = \begin{vmatrix} \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \\ \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{c} & \mathbf{c}.\mathbf{d} \\ \mathbf{d}.\mathbf{b} & \mathbf{d}.\mathbf{c} & \mathbf{d}.\mathbf{d} \end{vmatrix} = \tfrac{1}{2}a^6$$ (using A).

Again $|\mathbf{b} \times \mathbf{c}| = \frac{1}{2}\sqrt{3}a^2$, $|\mathbf{d} - \mathbf{b}|$ = length of one edge $= a$.

Putting these values in (1) we get $\sin \theta = \sqrt{2/3}$ and hence $\cos \theta = 1/\sqrt{3}$.

III. *First part.* $\overrightarrow{AC}.\overrightarrow{BD} = \mathbf{c}.(\mathbf{d} - \mathbf{b}) = \mathbf{c}.\mathbf{d} - \mathbf{c}.\mathbf{b} = 0$ in view of (A) *i.e.*, AC is perpendicular to BD. Similarly, we proceed for other pairs of opposite edges.

Second part. S.D. between AC and BD

$$=\frac{\mathbf{c}\times(\mathbf{d}-\mathbf{b}).(\mathbf{d}-\mathbf{c})}{|\mathbf{c}\times(\mathbf{d}-\mathbf{b})|}=\frac{[\mathbf{bcd}]}{|\mathbf{c}\times(\mathbf{d}-\mathbf{b})|}=\frac{1}{\sqrt{2}}a,$$

$=\frac{1}{2}\sqrt{2}a=\frac{1}{2}$ (diagonal of the square described on an edge).

Note that $|\mathbf{c}\times(\mathbf{d}-\mathbf{b})|=|\mathbf{c}|\ |\mathbf{d}-\mathbf{b}|\ \sin 90^\circ=a^2$.

IV. The equation of the plane ABC is $\mathbf{r}.\mathbf{b}\times\mathbf{c}=0$ (since it passes through the origin A and perpendicular to $\mathbf{b}\times\mathbf{c}$).

The distance of D from this plane is given by

$$\frac{\mathbf{d}.\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|}=\frac{[\mathbf{bcd}]}{|\mathbf{b}\times\mathbf{c}|}=\sqrt{\tfrac{2}{3}}a.$$

V. We leave it as an exercise for the students.

4·5. Equation of a Sphere.

Suppose we are required to find the equation of a sphere of centre C and radius a (Fig. 4.8). Let P be any point on its surface. With reference to an assigned origin O let the position vector of C and P be $\mathbf{c}$ and $\mathbf{r}$ respectively.

Evidently $\overrightarrow{CP}=\mathbf{r}-\mathbf{c}$, but $|\overrightarrow{CP}|=$ radius of the sphere $=a$.

Since the square of a vector is the square of its length, we may write

$$(\mathbf{r}-\mathbf{c})^2=a^2$$

i.e., $\mathbf{r}^2-2\mathbf{r}.\mathbf{c}+\mathbf{c}^2=a^2$

i.e., $\mathbf{r}^2-2\mathbf{r}.\mathbf{c}+k=o$ (1)

where $k=\mathbf{c}^2-a^2=c^2-a^2$.

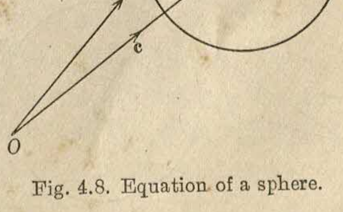


Fig. 4.8. Equation of a sphere.

Since the relation (1) is true for any point P on the surface of the sphere and by no others it represents the equation of the sphere relative to the origin O.

Particular cases.

I. When the origin O is a point on the surface of the sphere, $OC=a$ and hence $k=0$. The equation of the sphere will then be

$$\mathbf{r}^2-2\mathbf{r}.\mathbf{c}=0.$$

II. When O coincides with the centre C, we have simply $\overrightarrow{CP}=\mathbf{r}$ and the equation becomes $\mathbf{r}^2=a^2$.

Problems on Spheres.

(i) Intersection of a line with a sphere.

Consider a line $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ passing through a point $\mathbf{a}$ and parallel to the *unit vector* $\mathbf{b}$. If it cuts the sphere (1) then

$$(\mathbf{a}+t\mathbf{b})^2-2(\mathbf{a}+t\mathbf{b}).\mathbf{c}+k=0.$$

Hence the values of t corresponding to the points of intersection will be given by the roots of the quadratic equation

$$t^2+2\mathbf{b}.(\mathbf{a}-\mathbf{c})t+(\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k)=0 \qquad \cdots \quad (2)$$

Note that $\mathbf{b}^2=1$, since $\mathbf{b}$ is a unit vector. Otherwise the first term of the left side of (2) would be $t^2\mathbf{b}^2$. Now if the line cuts the sphere in two real and distinct points then we shall obtain two values t_1, t_2 for t, corresponding to which there are two points $P(\mathbf{a}+t_1\mathbf{b})$ and $Q(\mathbf{a}+t_2\mathbf{b})$ on the sphere (see Fig. 4.9). In this case we have

$$\{\mathbf{b}.(\mathbf{a}-\mathbf{c})\}^2 \geqslant \mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k.$$

Also we have $AP=t_1$, $AQ=t_2$, and $AP.AQ=t_1t_2=\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k$.

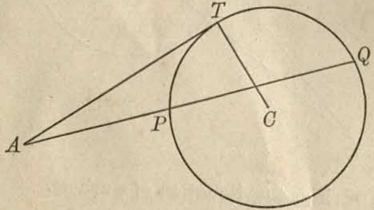


Fig. 4.9. Tangent plane

The last relation is independent of $\mathbf{b}$ and hence same for all straight lines through $\mathbf{a}$. If P and Q coincide at T on the sphere then we have the tangent AT to the sphere and

$$AT^2=\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k \qquad \cdots \quad (3)$$

We define the quantity $\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k$ which measures the square on the tangent from A to the surface of the sphere as the *power of a point* **a** *with respect to the sphere* (1). In particular, the power of the origin with respect to the sphere (1) is k. The tangents from A generates a cone, called *Tangent cone*, having its vertex at A and enveloping the sphere.

(ii) Equation of the tangent plane.

Suppose A lies on the sphere, then $AP=t_1=0$, *i.e.*, one root of (2) is zero so that

$$\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k=0 \qquad \cdots \quad (4)$$

In order that the line $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ through the point **a** may touch the surface of the sphere $AQ(=t_2)$ should also be equal to zero so that

$$\mathbf{b}.(\mathbf{a}-\mathbf{c})=0 \qquad \cdots \quad (5)$$

This incidentally shows that AC is perpendicular to the tangent line. The relations (4) and (5) are the *conditions of tangency*.

Now if **r** be any point on the tangent line then

$$(\mathbf{r}-\mathbf{a}).(\mathbf{a}-\mathbf{c})=0 \qquad \cdots \quad (6)$$

in view of (5). This shows that all tangents through A lie on a plane given by (6). This plane is known as *the tangent plane to the sphere* (1) *at* **a**. On simplification of (6) we get

$$\mathbf{r}.\mathbf{a}-\mathbf{a}^2-\mathbf{r}.\mathbf{c}+\mathbf{a}.\mathbf{c}=0 \qquad \cdots \quad (7)$$

Using the relation (4), we write (7) in the form

$$\mathbf{r}.\mathbf{a}-\mathbf{c}.(\mathbf{r}+\mathbf{a})+k=0, \qquad \cdots \quad (8)$$

which we take as the *standard equation of the tangent plane* at **a**.

(iii) Condition that a plane $\mathbf{r}.\mathbf{n}=p$ should touch the sphere (1).

Using the fact that the square of the perpendicular distance from the centre C on the plane (6) is equal to the square of the radius of the sphere it can be easily deduced that the condition of tangency is

$$\left(\frac{p-\mathbf{c}.\mathbf{n}}{|\mathbf{n}|^2}\right)^2=a^2 \qquad \cdots \quad (9)$$

(iv) Condition that the spheres

$$\mathbf{r}^2 - 2\mathbf{r}.\mathbf{c} + k = 0,\ \mathbf{r}^2 - 2\mathbf{r}.\mathbf{c}' + k' = 0 \qquad \cdots \quad (10)$$

should cut orthogonally is

$$2\mathbf{c}.\mathbf{c}' = k + k' \qquad \cdots \quad (11)$$

This is left as an exercise.

(v) Diametral plane.

The chord $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ ($\mathbf{b}$ = unit vector) of the sphere (1) is bisected at the point $\mathbf{a}$ if

$$(\mathbf{a} - \mathbf{c}).\mathbf{b} = 0$$

whence it is easy to see that the locus of the mid-points of chords of the sphere (1) parallel to the vector $\mathbf{b}$ is the plane

$$(\mathbf{r} - \mathbf{c}).\mathbf{b} = 0. \qquad \cdots \quad (12)$$

This plane is called the diametral plane bisecting chords parallel to $\mathbf{b}$.

(vi) Radical plane.

The *radical plane* of the two spheres given by (10) has the equation

$$2\mathbf{r}.(\mathbf{c} - \mathbf{c}') = k - k'.$$

We recall from Geometry that the locus of points whose powers with respect to two spheres are equal is a plane, called *radical plane*, perpendicular to the line joining their centres. Using this fact the result follows immediately.

(vii) Polar plane.

The polar plane of a point $\mathbf{a}$ with respect to the sphere (1) is

$$\mathbf{r}.\mathbf{a} - \mathbf{c}.(\mathbf{r} + \mathbf{a}) + k = 0 \qquad \cdots \quad (13)$$

If a variable line is drawn through A meeting the sphere in P and Q and if a point R is taken on this line such that A, R divide this line internally and externally in the same ratio, then the locus of R is a plane, called *polar plane of A with respect to*

the sphere. Now deduce (13) and also verify that if the polar plane of A passes through B, then the polar plane of B passes through A.

Examples. IV(H)

1. Show that the equation of the sphere described on the join of $\mathbf{a}$ and $\mathbf{b}$ as diameter is $(\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{b})=0$.

2. Show that any diameter of a sphere subtends a right angle at a point on the surface.

3. Show that the locus of a point, the sum of the squares on whose distances from n given points is constant, is a sphere.

4. O is any fixed point and P any point on given sphere. OP is joined and on it a point Q is taken so that $OP : OQ$ is a fixed ratio. Show that the locus of Q is a sphere.

5. In the previous example, if $OP.OQ$ = a constant quantity, the locus of Q is then called the *Inverse* of the sphere and O is called the *Centre of Inversion.* Find the inverse of a sphere *w.r.t.* a point (i) outside the sphere ; (ii) on its surface.

6. Prove that the locus of a point which moves so that its distances from two fixed points are in a constant ratio $k : 1$ is a sphere.

7. Obtain the centre of sphere inscribed in the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=a$.

8. Find the centre and radius of the sphere $\mathbf{r}^2-2\mathbf{r}.\mathbf{c}+h=0$.

Hints and Answers

1. Suppose $P(\mathbf{r})$ is any point on the surface of the sphere. Then

$$\overrightarrow{AP}=\mathbf{r}-\mathbf{a},\ \overrightarrow{BP}=\mathbf{r}-\mathbf{b}.$$

But diameter AB subtends a right angle at P. Hence

$$(\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{b})=0.$$

2. Let ACB be a diameter; C is the centre and A, B are points on the sphere (1). $\overrightarrow{CA} = -\overrightarrow{CB}$ *i.e.*, $\mathbf{a}+\mathbf{b}=2\mathbf{c}$, $\mathbf{a}$ is a point on the sphere and hence $\mathbf{a}^2-2\mathbf{a}.\mathbf{c}+k=0$. Now deduce $\mathbf{a}.\mathbf{b}=k$. Next if P be a point on the sphere then obtain (using these results)

$$\overrightarrow{PA}.\overrightarrow{PB}=0.$$

5. (i) A sphere; (ii) A plane.

7. Centre (x, y, z) is equidistant from the four planes and hence

$$x=y=z=\{a-(x+y+z)\}/\sqrt{3}.$$

Now obtain $x=y=z=\frac{a}{6}(3-\sqrt{3})$.

4·6. Applications to Mechanics.

I. Theorem of Rankine. *If four forces* $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ *acting at a point are in equilibrium then each force is proportional to the volume of the parallelopiped determined by the unit vectors in the directions of other three.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be the *unit* vectors in the directions of forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ and suppose their magnitudes are respectively P_1, P_2, P_3, P_4 so that

$$P_1\mathbf{a}=\mathbf{P}_1,\ P_2\mathbf{b}=\mathbf{P}_2, \text{ etc.}$$

Since the forces are in equilibrium their vector sum is zero.

Thus, $$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4=\mathbf{0}$$

i.e., $$P_1\mathbf{a}+P_2\mathbf{b}+P_3\mathbf{c}+P_4\mathbf{d}=\mathbf{0} \qquad \cdots \quad (1)$$

Forming successively the scalar product of (1) with $\mathbf{c}\times\mathbf{d}$, $\mathbf{a}\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b}$, we obtain

$$\left.\begin{aligned} P_1[\mathbf{acd}]+P_2[\mathbf{bcd}]&=0\\ P_2[\mathbf{bac}]+P_4[\mathbf{dac}]&=0\\ P_3[\mathbf{cab}]+P_4[\mathbf{dab}]&=0 \end{aligned}\right\} \qquad \cdots \quad (2)$$

With proper choice of signs we can now deduce from (2)

$$\frac{P_1}{[\mathbf{bcd}]}=\frac{P_2}{[\mathbf{cad}]}=\frac{P_3}{[\mathbf{abd}]}=\frac{P_4}{[\mathbf{bac}]}.$$

Thus each force is proportional to the scalar triple product of unit vectors in the directions of the other three forces and hence to the volume of the parallelopiped determined by them.

Corollary. *Lami's Theorem*: If three forces acting at a point are in equilibrium then each force is proportional to the sine of the angle between the other two.

In this case $P_1\mathbf{a}+P_2\mathbf{b}+P_3\mathbf{c}=\mathbf{0}$. Take successively the vector product with **a**, **b**, **c**; the result will follow.

II. Work done by a force.

DEFINITION. When the point of application of a force **F** experiences a displacement represented by the vector **d**, the work done by the force is the product of the displacement by the component of the force in the direction of the displacement.

Thus

$$\text{Work done}=|\mathbf{d}|\ |\mathbf{F}|\cos\theta=\mathbf{F}.\mathbf{d},$$

where θ is the angle between **F** and **d**.

No work is said to be done if **d** is perpendicular to **F**.

If $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$ is the resultant of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on a particle subjected to a displacement **d**, then

total work done by the two forces

$$=\mathbf{F}_1.\mathbf{d}+\mathbf{F}_2.\mathbf{d}=(\mathbf{F}_1+\mathbf{F}_2).\mathbf{d}\quad\text{(Distributive Law)}$$

$$=\mathbf{F}.\mathbf{d}=\text{work done by the resultant force }\mathbf{F}.$$

That is, work of the resultant is equal to the sum of the works done by the separate forces. We can easily generalise this result.

Again, when the point of application of a force **F** experiences two successive displacements (Fig. 4.10)

$$\mathbf{d}_1=\overrightarrow{AB}\text{ and }\mathbf{d}_2=\overrightarrow{BC}$$

then total work done $=\mathbf{F}.\mathbf{d}_1+\mathbf{F}.\mathbf{d}_2=\mathbf{F}.(\mathbf{d}_1+\mathbf{d}_2).=\mathbf{F}.\overrightarrow{AC}.$

That is, the sum of the works in two displacements is equal to the work done by **F** in the single displacement $\mathbf{d}_1 + \mathbf{d}_2$. In a similar manner we may prove :

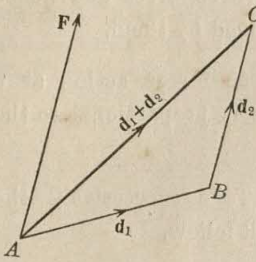


Fig. 4.10
Works done by several displacements

The total work done by a constant force when its point of application experiences any finite number of successive displacements, is equal to the work done by the force in the single displacement from the beginning to the end.

In case, the point of application traverses consecutively the sides of a closed polygon the total work done is zero.

III. (A) Moment of a force about a point.

In art. 3·3 (Page 97) we have already introduced this concept.

Let **F** be a force *applied to a body at the point P* (Fig. 3.6). If **r** be the vector from any point O to P, then we know that the vector

$$\mathbf{m} = \mathbf{r} \times \mathbf{F}$$

is called the *moment vector* of the force **F** about the point O ; **m** is also called the *torque* of the force **F** about O. **m** is evidently perpendicular to the plane containing O and the line of **F**. In magnitude, it is equal to the product of the force by the perpendicular distance ON from O to the line of **F**. Conversely, given F and **m** we may specify the force completely (art. 3·3).

Important observations.

1. The vector **m** does not depend on the choice of the point P on the line of **F**. Thus if Q be any other point on the line of **F** we have

$$\overrightarrow{OQ} \times \mathbf{F} = (\overrightarrow{OP} + \overrightarrow{PQ}) \times \mathbf{F} = \overrightarrow{OP} \times \mathbf{F} \quad (\because \ \overrightarrow{PQ} \times \mathbf{F} = \mathbf{0})$$

But the vector **m** depends on the choice of O.

2. $\mathbf{m}.\mathbf{F} = \mathbf{r} \times \mathbf{F}.\ \mathbf{F} = 0$ suggests that $\mathbf{F}$ and $\mathbf{m}$ are perpendicular to each other.

3. If the force $\mathbf{F}$ is the resultant of a number of concurrent forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots\ldots \mathbf{F}_n$ then

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots\cdots \mathbf{F}_n)$$
$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \mathbf{r} \times \mathbf{F}_3 + \cdots + \mathbf{r} \times \mathbf{F}_n \quad \text{(Distributive Law)}$$

That is, the moment of the resultant of any number of forces about a point is equal to the sum of their separate moments (Generalised Varignon's Theorem).

(B) Moment of a force about a line.

Suppose we require to find the moment of $\mathbf{F}$ about a line whose direction is given by the *unit* vector $\mathbf{b}$ (see Fig. 4.11). We take *any point* O on that line and find the moment vector $\mathbf{m} = \mathbf{r} \times \mathbf{F}$ of the force $\mathbf{F}$ about O. We now define the projection of $\mathbf{m}$ on $\mathbf{b}$ as the moment of the force $\mathbf{F}$ about the line $\mathbf{b}$. Thus the moment of $\mathbf{F}$ about the line $\mathbf{b}$ is given by

$$\mathbf{m}.\mathbf{b} = \mathbf{r} \times \mathbf{F}.\mathbf{b},$$

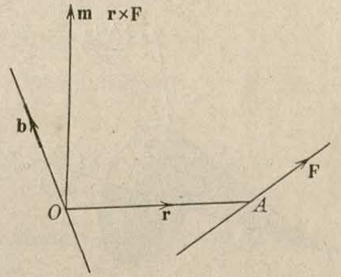


Fig. 4.11. Moment of a force about a line

which is six times the volume of a tetrahedron constructed with line of $\mathbf{F}$ and any segment of unit length $\mathbf{b}$ as a pair of opposite edges.

It follows that if α be the angle between the lines of $\mathbf{F}$ and $\mathbf{b}$ and d be the length of S.D. between them then the moment of $\mathbf{F}$ about the line $\mathbf{b}$ is $|\mathbf{F}|\ d \sin \alpha$.

Moments about origin and about the axes of coördinates.

Expressing the vectors $\mathbf{F}$, $\mathbf{r}$, $\mathbf{m}$ in terms of their rectangular components as

$$\mathbf{F} = (X, Y, Z),\ \mathbf{r} = (x, y, z),\ \mathbf{m} = (m_x, m_y, m_z),$$

the moment of $\mathbf{F}$ about $O = \mathbf{m} = \mathbf{r} \times \mathbf{F}$

i.e., $(m_x, m_y, m_z) = (yZ - zY, zX - xZ, xY - yX)$

so that $m_x = yZ - zY$, $m_y = zX - xZ$, $m_z = xY - yX$.

The moment of $\mathbf{F}$ about the X-axis

$\mathbf{m}.\mathbf{i} = yZ - zY = m_x$, since the unit vector along X-axis is $\mathbf{i}$.

Similarly we can find the moments about the other two axes.

Thus the moment of a force F about any axis through O is the resolved part of moment of F about O along this axis.

IV. Rotation about a fixed axis : Motion of a rigid body.

We consider the motion of a rigid body about a fixed axis ON. Let ϕ be the angle at any time t between two planes through ON, one of which is fixed in the body and the other fixed in space. Then $\omega = d\phi/dt$ is called the *angular speed* of the body about ON. If $\mathbf{k}$ denote the unit vector along the axis of rotation, then the vector $\mathbf{A} = \omega\mathbf{k}$ is called the *angular velocity* of the body about ON.

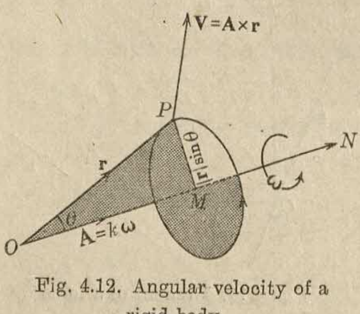


Fig. 4.12. Angular velocity of a rigid body

Velocity of any point P of the body can be expressed in terms of $\mathbf{A}$. Suppose P is *any point* of the body (Fig. 4.12). Let $\mathbf{r}$ be the vector directed from O to P. Draw PM perpendicular to ON. Then P describes a circle whose centre is M, radius is PM and the plane is perpendicular to the axis ON. Then if $\mathbf{v}$ denotes the velocity of P, then (assuming $\angle PON = \theta$),

(i) *magnitude* of $\mathbf{v} = |\mathbf{v}| = \omega PM = \omega|\mathbf{r}| \sin\theta = |\mathbf{A} \times \mathbf{r}|$

(ii) *direction* of $\mathbf{v}$ is normal to the plane OMP.

(iii) *sense* of $\mathbf{v}$ is such that $\mathbf{A}, \mathbf{r}, \mathbf{v}$ form a right handed system.

In other words, $\mathbf{v} = \mathbf{A} \times \mathbf{r}$.

That is, the velocity of any particle of a body revolving about a fixed axis is equal to the vector product of angular velocity and the position vector of the particle referred to any origin on the axis.

4·61. Illustrative Examples.

1. A particle, acted on by constant forces (4, 1, −3) and (3, 1, −1), is displaced from the point A (1, 2, 3) to the point B (5, 4, 1). Find the work done by the forces on the particle.

Solution. We call the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ so that

$$\mathbf{F}_1=(4,\ 1,\ -3),\ \ \mathbf{F}_2=(3,\ 1,\ -1).$$

Their resultant $\mathbf{F}$ is given by

$$\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2=(4, 1,\ -3)+(3,\ 1,\ -1)=(7,\ 2,\ -4).$$

The displacement vector $\mathbf{d}=\overrightarrow{AB}=(5,\ 4,\ 1)-(1,\ 2,\ 3)=(4,\ 2,\ -2).$

$\therefore$ Work done by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ = Work done by $\mathbf{F}$

$$=\mathbf{F}.\mathbf{d}=(7,\ 2,\ -4).(4,\ 2,\ -2)$$

$$=40 \text{ units.}$$

2. A force $3\mathbf{i}+\mathbf{k}$ acts through the point $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$. Find the torque about the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Solution. Let P be the point $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and O be the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$. Then $\overrightarrow{OP}=\mathbf{r}=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})-(\mathbf{i}+2\mathbf{j}-\mathbf{k})=\mathbf{i}-3\mathbf{j}+4\mathbf{k}.$

Since the force vector $\mathbf{F}=3\mathbf{i}+\mathbf{k}$, the required torque is

$$\mathbf{r}\times\mathbf{F}=-3\mathbf{i}+11\mathbf{j}+9\mathbf{k}.$$

3. Find the moment of the force $\overrightarrow{PQ}$ about an axis through A (3, 1, 0) in the direction of the vector (3, 4, 12) : Given that the position vectors of P and Q are (1, 3, 1) and (3, 5, 2) respectively.

Solution. $\overrightarrow{PQ}=(3,\ 5,\ 2)-(1,\ 3,\ 1)=(2,\ 2,\ 1)$; $\overrightarrow{AP}=(-2,\ 2,\ 1).$

Moment of $\overrightarrow{PQ}$ about $A=\overrightarrow{AP}\times\overrightarrow{PQ}$

$$=(-2,\ 2,\ 1)\times(2,\ 2,\ 1)=(0,\ 4,\ -8).$$

$\therefore$ Moment of $\overrightarrow{PQ}$ about the given axis

$$=(0,\ 4,\ -8)\,.\,\tfrac{1}{13}(3,\ 4,\ -12)=-80/13.$$

4. A rigid body is spinning with an angular speed of 4 radians per second about an axis through O (1, 3, -1) in the direction of the vector (0, 3, -1). Find the velocity of any point P (4, -2, 1) on the body.

Solution. Here $\overrightarrow{OP}=(4, -2, 1)-(1, 3, -1)=(3, -5, 2)$.

$$\text{angular velocity } \mathbf{A}=4\cdot\frac{1}{\sqrt{10}}(0, 3, -1)$$

$$\therefore \text{ Velocity of } P=\mathbf{A}\times\overrightarrow{OP}$$

$$=\frac{4}{\sqrt{10}}(0, 3, -1)\times(3, -5, 2)$$

$$=\frac{4}{\sqrt{10}}(1, -3, -9).$$

This gives a speed of $4\sqrt{(91/10)}=12$ (approx) radians per second in the direction of the vector $(1, -3, -9)$.

Examples. IV(I)

1. The point of application of a force $\mathbf{F}=(5, 10, 15)$ pounds is displaced from the point A (1, 0, 3) to the point B (3, -1, -6). Show that the work done by the force is 135 foot pound units, the unit of length being 1 foot.

2. A force of 15 units acts in the direction of the vector $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and passes through a point $2\mathbf{i}-2\mathbf{j}+2\mathbf{k}$. Find the moment of the force about the point $\mathbf{i}+\mathbf{j}+\mathbf{k}$. [$15\sqrt{13}$]

3. A rigid body is spinning with angular speed of 27 radians per second about an axis parallel to the direction of the vector (2, 1, -2) and passing through the point (1, 3, -1). Show that the velocity of a point of the body whose position vector is (4, 8, 1) is 9 (12, -10, 7).

4. Find the moment about the point $3\mathbf{i}+\mathbf{j}$ and moment about the line $\mathbf{r}=(3-4\lambda)\mathbf{i}+\mathbf{j}+3\lambda\mathbf{k}$ of the force $7\mathbf{i}-\mathbf{j}+2\mathbf{k}$ acting at the point $6\mathbf{j}-\mathbf{k}$. [$9\mathbf{i}-\mathbf{j}+32\mathbf{k}$, $-132/5$]

5. Show that the torque about the point A (3, -1, 3) of a force (4, 2, 1) through B (5, 2, 4) is (1, 2, -8).

5

Differentiation and Integration of Vectors

5·1. Functions of a single scalar variable : Limit and continuity.

Vector Function.

If to each value of a scalar variable t, in some interval (a, b), there corresponds a unique vector $\mathbf{F}$, determined by any law whatsoever, then $\mathbf{F}$ is called a *vector function* of t, a relation indicated by $\mathbf{F}(t)$.

Illustration. Let $\mathbf{r} = \overrightarrow{OP}$ be the vector from a fixed origin O to a variable point P on a curve in space. Suppose there exists an independent scalar variable t such that when the value of t is given, the terminus of $\mathbf{r}$ can be located. We then say that $\mathbf{r}$ is a vector function of t.

Note. If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three unit vectors along three fixed directions it is possible to decompose $\mathbf{F}(t)$ as

$$\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k},$$

where $F_1(t)$, $F_2(t)$ and $F_3(t)$ are scalar functions of t. We may indicate such a relation by

$$\mathbf{F} = (F_1, F_2, F_3).$$

Limit of a Vector Function.

Roughly speaking, a function $\mathbf{F}(t)$ is said to approach a constant vector $\mathbf{A}$ as limit, when t approaches t_0, if the *length of the vector* $\mathbf{F}(t) - \mathbf{A}$ approaches zero.

To make the idea clear, first let $\mathbf{A} \neq \mathbf{0}$ then $\mathbf{F}(t)$ has almost the same direction and almost the same magnitude as that of $\mathbf{A}$ when t is near t_0 ; but if $\mathbf{A} = \mathbf{0}$ then the direction of $\mathbf{F}(t)$ may

vary arbitrarily provided merely that the magnitude $|\mathbf{F}(t)|$ approaches zero. *In symbols*, we write the above statements as

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{A}.$$

Analytical Definition.

A vector function $\mathbf{F}(t)$ is said to tend to a limit $\mathbf{A}$, as t approaches t_0, if, to any preassigned positive number ε, however small, there corresponds a positive number δ, such that

$$|\mathbf{F}(t) - \mathbf{A}| < \varepsilon \qquad \text{when } 0 < |t - t_0| \leqslant \delta$$

We express this fact by writing

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{A}.$$

Theorems on Limit.

We state the following results without proof. The curious reader may, however, attempt to prove them by using the analytical definition given above ; the line of arguments will be almost similar to the corresponding results in Scalar Calculus (See *Maity* and *Ghosh*, *Differential Calculus* (1961) chapter 4).

Theorem 1. *Suppose*

$$\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k} \text{ and } \mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *are unit vectors in three fixed mutually perpendicular directions.*

Then $\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{A}$

implies three relations :

(i) $\lim_{t \to t_0} F_1(t) = A_1$; (ii) $\lim_{t \to t_0} F_2(t) = A_2$; (iii) $\lim_{t \to t_0} F_3(t) = A_3$

and conversely.

Theorem 2. *If* $\mathbf{F}(t)$ *and* $\mathbf{f}(t)$ *are two vector funstions of the scalar variable* t, *then*

(i) $\lim_{t \to t_0} [\mathbf{F}(t) \pm \mathbf{f}(t)] = \lim_{t \to t_0} \mathbf{F}(t) \pm \lim_{t \to t_0} \mathbf{f}(t)$;

(ii) $\lim\limits_{t \to t_0} [\mathbf{F}(t) . \mathbf{f}(t)] = \lim\limits_{t \to t_0} \mathbf{F}(t) . \lim\limits_{t \to t_0} \mathbf{f}(t)$;

(iii) $\lim\limits_{t \to t_0} [\mathbf{F}(t) \times \mathbf{f}(t)] = \lim\limits_{t \to t_0} \mathbf{F}(t) \times \lim\limits_{t \to t_0} \mathbf{f}(t)$;

Also if $\phi(t)$ *be a scalar function of* t *then*

(iv) $\lim\limits_{t \to t_0} [\phi(t)\mathbf{F}(t)] = \lim\limits_{t \to t_0} \phi(t) \lim\limits_{t \to t_0} \mathbf{F}(t)$;

provided all the limits under consideration exist.

Continuity.

DEFINITION. Let $\mathbf{F}(t)$ denote a vector function of a scalar variable t over the interval $a \leqslant t \leqslant b$. Let t_0 be a value of t in the given interval. Then $\mathbf{F}(t)$ is said to be continuous at $t = t_0$, if,

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{F}(t_0).$$

Also $\mathbf{F}(t)$ is said to be continuous in (a, b) if it is continuous for every value of this interval.

Theorem on Continuity.

The sum, difference, scalar product and vector product of two continuous functions are also continuous and the product of a continuous vector function with a continuous scalar function is also continuous.

Proof. The results follow from Theorem 2 on limits. We take one case. Suppose $\mathbf{F}(t)$ and $\mathbf{f}(t)$ are continuous vector functions of t for $t = t_0$, then we have

$$\begin{aligned}\lim_{t \to t_0} [\mathbf{F}(t) + \mathbf{f}(t)] &= \lim_{t \to t_0} \mathbf{F}(t) + \lim_{t \to t_0} \mathbf{f}(t) \\ &= \mathbf{F}(t_0) + \mathbf{f}(t_0)\end{aligned}$$

That is, $\mathbf{F}(t) + \mathbf{f}(t)$ is continuous at $t = t_0$.

The other results can be similarly proved.

With the help of Theorem 1 we see that if

$$\mathbf{F}(t) = [F_1(t), F_2(t), F_3(t)]$$

be continuous, then $F_1(t)$, $F_2(t)$ and $F_3(t)$ are also continuous scalar functions and conversely.

5·2. Derivative of a vector.

Suppose that $\mathbf{F}(t)$ is a single-valued function of a scalar variable t. Relative to a fixed origin O let $\overrightarrow{OP}$ be the value $\mathbf{F}(t)$ corresponding to some fixed value t, say $t=t_0$, of the scalar variable. Now suppose the value $t_0+\Delta t$ of t corresponds to the value $\mathbf{F}(t_0+\Delta t)$ of the vector function and let $\overrightarrow{OQ}=\mathbf{F}(t_0+\Delta t)$ (Fig. 5.1). The increment in $\mathbf{F}(t)$ when t changes from t_0 to $t_0+\Delta t$ is

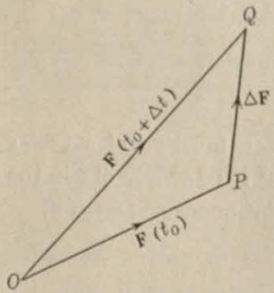


Fig. 5.1
Increment of a vector

$$\Delta \mathbf{F}=\mathbf{F}(t_0+\Delta t)-\mathbf{F}(t_0)$$
$$=\overrightarrow{OQ}-\overrightarrow{OP}=\overrightarrow{PQ}.$$

Since Δt is a scalar and $\Delta \mathbf{F}$ is a vector, we may now obtain

$$\frac{\Delta \mathbf{F}}{\Delta t}=\frac{\overrightarrow{PQ}}{\Delta t}.$$

This is itself a vector extending along PQ. If this vector approaches an *unique finite* limit when Δt approaches zero, that limit, is called the *derivative* of $\mathbf{F}(t)$ with respect to t at $t=t_0$.

When this limit exists, the function $\mathbf{F}(t)$ is said to be *derivable* at $t=t_0$ and this process of determining the derivative of the function is called *differentiation* or *derivation*. Thus the derivative of $\mathbf{F}(t)$ with respect to t at $t=t_0$, when exists, is givén by.

$$\lim_{\Delta t\to 0}\frac{\Delta \mathbf{F}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\mathbf{F}(t_0+\Delta t)-\mathbf{F}(t_0)}{\Delta t}=\frac{d\mathbf{F}}{dt}=\mathbf{F}'(t_0).$$

If at every point in the interval of definition of the function $\mathbf{F}(t)$ the derivative exists we obtain, the derived function

$$\mathbf{F}'(t) \text{ or } \frac{d\mathbf{F}}{dt}$$

in this interval, where

$$\mathbf{F}'(t)=\lim_{\Delta t\to 0}\frac{\mathbf{F}(t+\Delta t)-\mathbf{F}(t)}{\Delta t},$$

t being any point in the interval of definition.

Relation between Continuity and Derivability.

Theorem. *If* $\mathbf{F}'(t)$ *exists at* $t=t_o$, *then* $\mathbf{F}(t)$ *is continuous at* $t=t_o$. *In other words, every derivable function is continuous.*

Proof. Clearly, $\lim\limits_{\Delta t\to 0}\,[\mathbf{F}(t_o+\Delta t)-\mathbf{F}(t_o)]$

$$=\lim_{\Delta t\to 0}\left[\Delta t\left\{\frac{\mathbf{F}(t_o+\Delta t)-\mathbf{F}(t_o)}{\Delta t}\right\}\right]$$

$$=\left\{\lim_{\Delta t\to 0}\,\Delta t\right\}\left\{\lim_{\Delta t\to 0}\,\frac{\mathbf{F}(t_o+\Delta t)-\mathbf{F}(t_o)}{\Delta t}\right\}=0.\mathbf{F}'(t_o)=0.$$

$$\therefore\ \lim_{\Delta t\to 0}\,\mathbf{F}(t_o+\Delta t)=\mathbf{F}(t_o)$$

proving that $\mathbf{F}(t)$ is continuous at $t=t_o$.

Otherwise. If $\mathbf{F}'(t_o)$ exists we may write

$$\frac{\mathbf{F}(t_o+\Delta t)-\mathbf{F}(t_o)}{\Delta t}=\mathbf{F}'(t_o)+\varepsilon$$

where $\varepsilon\to 0$ as $\Delta t\to 0$; hence

$$\mathbf{F}(t_o+\Delta t)-\mathbf{F}(t_o)=\{\mathbf{F}'(t_o)+\varepsilon\}\Delta t$$

and as $\Delta t\to 0$,

$$\mathbf{F}(t_o+\Delta t)\to\mathbf{F}(t_o).$$

Hence etc.

Converse. The converse of this theorem is not true. Thus

$$\mathbf{F}(t)=|t|\,\mathbf{i},$$

is continuous at $t=0$ but not derivable there.

For, $\quad \left|\mathbf{F}(t)-\mathbf{F}(0)\right|=\left||t|\,\mathbf{i}-0\right|=|t|$

whence, $\quad \lim\limits_{t\to 0}\mathbf{F}(t)=\mathbf{0}=\mathbf{F}(0).$

Also $$\frac{\mathbf{F}(t)-\mathbf{F}(0)}{t-0}=\frac{|t|\,\mathbf{i}}{t}$$

so that the limit is $\mathbf{i}$ or $-\mathbf{i}$ according as t tends to zero through positive or through negative values. Hence $\mathbf{F}'(0)$ does not exist since the limit is not unique.

Derivatives of higher orders.

If the derived function $\mathbf{F}'(t)$ is also derivable *i.e.*, if

$$\lim_{\Delta t\to 0}\frac{\mathbf{F}'(t+\Delta t)-\mathbf{F}'(t)}{\Delta t}$$

exists then we say that the second derivative of $\mathbf{F}(t)$ exists and we denote this second derivative by

$$\frac{d}{dt}\left(\frac{d\mathbf{F}}{dt}\right) \text{ or } \frac{d^2\mathbf{F}}{dt^2} \text{ or } \mathbf{F}''(t).$$

Higher order derivatives

$$\frac{d^3\mathbf{F}}{dt^3},\ \frac{d^4\mathbf{F}}{dt^4},\ \text{etc.}$$

are similarly defined as in scalar calculus.

5·3. Derivatives of sums and products.

The fundamental rules of differentiations hold for vectors very much like scalar calculus, but with this important difference that the order of the factors must remain unchanged in all expressions where a change in the order of the vectors, would make a different significance, as in cases of cross products.

Law I. *The components of* $\mathbf{F}'(t)$ *are derivatives of the components of* $\mathbf{F}(t)$. *Thus, if*

$$\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$$

is a derivable (vector) function of t then F_1, F_2, F_3 *are also derivable (scalar) functions of t and further*

$$\frac{d\mathbf{F}}{dt}=\frac{dF_1}{dt}\,\mathbf{i}+\frac{dF_2}{dt}\,\mathbf{j}+\frac{dF_3}{dt}\,\mathbf{k}$$

and conversely.

Proof. We have

$$\frac{\mathbf{F}(t+\triangle t)-\mathbf{F}(t)}{\triangle t}=\frac{F_1(t+\triangle t)-F_1(t)}{\triangle t}\,\mathbf{i}$$
$$+\frac{F_2(t+\triangle t)-F_2(t)}{\triangle t}\,\mathbf{j}+\frac{F_3(t+\triangle t)-F_3(t)}{\triangle t}\,\mathbf{k}.$$

By Theorem 1 of art. 5·1, the limit on the left (as $\triangle t \to 0$) exists if any only if the three limits

$$\lim_{\triangle t\to 0}\frac{F_1(t+\triangle t)+F_1(t)}{\triangle t},\ \text{etc.}$$

all exist. When these three limits exist we obtain easily

$$\mathbf{F}'(t)=F'_1(t)\,\mathbf{i}+F'_2(t)\,\mathbf{j}+F'_3(t)\,\mathbf{k}.$$

Hence the proposition.

Law II. *The derivative of any constant vector* $\mathbf{c}$ *is zero and conversely. That is,*

$$\frac{d\mathbf{c}}{dt}=\mathbf{0}$$

where $\mathbf{c}$ *is a constant vector and conversely if*

$$\frac{d\mathbf{c}}{dt}=\mathbf{0}$$

then $\mathbf{c}$ *is a constant vector.*

The proof is obvious, for an increment $\triangle t$ in the scalar variable t produces no change in $\mathbf{c}$ *i.e.* $\triangle\mathbf{c}=\mathbf{0}$ so that $\triangle\mathbf{c}/\triangle t=\mathbf{0}$ for every increment $\triangle t$ and consequently

$$\frac{d\mathbf{c}}{dt}=\mathbf{0}.$$

The converse is also true.

Suppose the derivative of $\mathbf{c}$ is zero. Then if $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, we have, by Law I,

$$\frac{d\mathbf{c}}{dt}=\frac{dc_1}{dt}\,\mathbf{i}+\frac{dc_2}{dt}\,\mathbf{j}+\frac{dc_3}{dt}\,\mathbf{k}.$$

Now $\quad \dfrac{d\mathbf{c}}{dt}=\mathbf{0}$ implies $\dfrac{dc_1}{dt}=0,\ \dfrac{dc_2}{dt}=0,\ \dfrac{dc_3}{dt}=0,$

whence it follows that c_1, c_2, c_3 are constants so that the vector $\mathbf{c}$ is a constant vector.

Law III. *The derivative of the sum of two derivable vector functions* $\mathbf{u}$ *and* $\mathbf{v}$ *of the scalar variable t, is equal to the sum of their derivatives. That is,*

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v})=\frac{d\mathbf{u}}{dt}+\frac{d\mathbf{v}}{dt}.$$

Some authors prefer dashes to indicate derivations : accordingly,

$$(\mathbf{u}+\mathbf{v})'=\mathbf{u}'+\mathbf{v}'.$$

Proof. The increment in $\mathbf{u}+\mathbf{v}=\triangle(\mathbf{u}+\mathbf{v})$

$$=(\mathbf{u}+\triangle\mathbf{u}+\mathbf{v}+\triangle\mathbf{v})-(\mathbf{u}+\mathbf{v})$$

$$=\triangle\mathbf{u}+\triangle\mathbf{v}$$

$$\therefore \quad \frac{\triangle(\mathbf{u}+\mathbf{v})}{\triangle t}=\frac{\triangle\mathbf{u}}{\triangle t}+\frac{\triangle\mathbf{v}}{\triangle t}.$$

Proceeding to the limit as $\triangle t \to 0$, we have

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v})=\frac{d}{dt}\mathbf{u}+\frac{d}{dt}\mathbf{v}\cdot$$

The rule can obviously be extended for any finite number of vector functions. Similarly it can be shown that

$$\frac{d}{dt}(\mathbf{u}-\mathbf{v})=\frac{d\mathbf{u}}{dt}-\frac{d\mathbf{v}}{dt}.$$

Law IV. *The derivative of the product $f\,\mathbf{u}$ of a scalar f and a vector* $\mathbf{u}$, *both being derivable functions of the scalar varible t, is given by*

$$\frac{d}{dt}(f\mathbf{u})=f\frac{d\mathbf{u}}{dt}+\frac{df}{dt}\mathbf{u}$$

$$\text{or} \quad (f\mathbf{u})'=f\mathbf{u}'+f'\mathbf{u}.$$

Proof. If $\triangle f$ and $\triangle\mathbf{u}$ are the increments of f and $\mathbf{u}$ corres-

ponding to the increment $\triangle t$, then the increment in their product is given by

$$\triangle(f\mathbf{u}) = (f + \triangle f)(\mathbf{u} + \triangle \mathbf{u}) - f\mathbf{u}$$

$$= (\triangle f)\mathbf{u} + f(\triangle \mathbf{u}) + (\triangle f)(\triangle \mathbf{u})$$

$$\therefore \quad \frac{\triangle(f\mathbf{u})}{\triangle t} = \frac{\triangle f}{\triangle t}\mathbf{u} + f\frac{\triangle \mathbf{u}}{\triangle t} + \frac{\triangle f}{\triangle t}\triangle \mathbf{u}.$$

Proceeding to the limit as $\triangle t$ tends to zero the result follows.

Note. Law I can be deduced by applying Law IV in

$$\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k}$$

and remembering that the derivative of constant vectors **i**, **j**, **k** all will be zero.

Law V. *The derivatives of the scalar and vector products* $\mathbf{u}.\mathbf{v}$ *and* $\mathbf{u} \times \mathbf{v}$ *are given by the formulae*

$$\frac{d}{dt}(\mathbf{u}.\mathbf{v}) = \mathbf{u}\,.\,\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}.\,\mathbf{v} \qquad (A)$$

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v} \qquad (B)$$

Their proofs depend essentially on the distributive laws for the dot and cross products.

CAUTION. In (B) the *order* of the factors must be maintained. We give the proof of (B) and leave the proof of (A) to the students.

Proof of (B). If $\mathbf{u}$ and $\mathbf{v}$ receive increments $\triangle \mathbf{u}$ and $\triangle \mathbf{v}$ for an increment $\triangle t$ in t, then the increment in $\mathbf{u} \times \mathbf{v}$ is given by

$$\triangle(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} + \triangle \mathbf{u}) \times (\mathbf{v} + \triangle \mathbf{v}) - \mathbf{u} \times \mathbf{v}$$

$$= \mathbf{u} \times \triangle \mathbf{v} + \triangle \mathbf{u} \times \mathbf{v} + \triangle \mathbf{u} \times \triangle \mathbf{v}$$

whence,
$$\frac{\triangle(\mathbf{u} \times \mathbf{v})}{\triangle t} = \mathbf{u} \times \frac{\triangle \mathbf{v}}{\triangle t} + \frac{\triangle \mathbf{u}}{\triangle t} \times \mathbf{v} + \frac{\triangle \mathbf{u}}{\triangle t} \times \triangle \mathbf{v}$$

and in the limit

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}.$$

Note 1. We have assumed that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ are all derivable functions of t.

Note 2. In both the terms on the right side of the formula (B), $\mathbf{u}$ and $\mathbf{v}$ occupy the same positions as they do in the product $\mathbf{u}\times\mathbf{v}$. It would not be correct to replace the second term by $\mathbf{v}\times\dfrac{d\mathbf{u}}{dt}$ unless the sign is changed at the same time.

An Important Corollary. In (A) put $\mathbf{v}=\mathbf{u}$, we have then the useful formula :

$$\frac{d}{dt}(\mathbf{u}.\mathbf{u})=\frac{d}{dt}(\mathbf{u}^2)=2\mathbf{u}.\frac{d\mathbf{u}}{dt}.$$

In view of the fact $\mathbf{u}.\mathbf{u}=\mathbf{u}^2=|\mathbf{u}|^2$ and the derivative of the scalar $|\mathbf{u}|^2$ is

$$2|\mathbf{u}|\frac{d}{dt}\{|\mathbf{u}|\}$$

we have, $\quad \mathbf{u}.\dfrac{d\mathbf{u}}{dt}=|\mathbf{u}|\dfrac{d}{dt}\{|\mathbf{u}|\}.$

Again, if we put $\mathbf{v}=\dfrac{d\mathbf{u}}{dt}$ in (B) we get

$$\frac{d}{dt}\left(\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right)=\frac{d\mathbf{u}}{dt}\times\frac{d\mathbf{u}}{dt}+\mathbf{u}\times\frac{d^2\mathbf{u}}{dt^2}=\mathbf{u}\times\frac{d^2\mathbf{u}}{dt^2}.$$

Law VI. *If* $\mathbf{u}$ *is a derivable function of a scalar variable* s, *and* s *is again a derivable function of another scalar variable* t, *then*

$$\frac{d\mathbf{u}}{dt}=\frac{d\mathbf{u}}{ds}\frac{ds}{dt}$$

Proof. Left as an exercise.

5·31. Two important theorems.

Theorem 1. Vectors with constant magnitude.

A necessary and sufficient condition that a proper vector $\mathbf{u}$ *has a constant length is that*

$$\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}=0 \qquad \dots \qquad \text{(i)}$$

Proof. Since $|\mathbf{u}|^2 = \mathbf{u}^2 = \mathbf{u}.\mathbf{u}$, we have from Law IV

$$\frac{d}{dt}|\mathbf{u}|^2 = \frac{d}{dt}(\mathbf{u}^2) = 2\mathbf{u}\,.\,\frac{d\mathbf{u}}{dt}.$$

Now $|\mathbf{u}|$ = constant implies (i) and conversely. Hence the theorem.

We note that the theorem expresses the fact that the *derivative of a vector of constant length is perpendicular to that vector.*

Theorem 2. Vectors with constant direction.

A necessary and sufficient condition that a proper vector $\mathbf{u}$ *always remains parallel to a fixed line is that*

$$\mathbf{u} \times \frac{d\mathbf{u}}{dt} = \mathbf{0} \qquad \cdots \qquad \text{(ii)}$$

Proof. Let $\mathbf{u} = u\,\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is the unit vector in the direction of $\mathbf{u}$ and u be its length.

Then,

$$\mathbf{u} \times \frac{d\mathbf{u}}{dt} = u\hat{\mathbf{u}} \times \frac{d}{dt}(u\hat{\mathbf{u}})$$

$$= u\hat{\mathbf{u}} \times \left(\frac{du}{dt}\hat{\mathbf{u}} + u\frac{d\hat{\mathbf{u}}}{dt}\right) = u^2\hat{\mathbf{u}} \times \frac{d\hat{\mathbf{u}}}{dt} \quad (\because\ \hat{\mathbf{u}} \times \hat{\mathbf{u}} = \mathbf{0})$$

When $\mathbf{u}$ remains parallel to a fixed line, $\hat{\mathbf{u}}$ is constant and hence

$$\frac{d\hat{\mathbf{u}}}{dt} = \mathbf{0}$$

and as such the condition follows necessarily.

Conversely, since $\mathbf{u} \neq \mathbf{0}$ the given condition leads to

$$\hat{\mathbf{u}} \times \frac{d\hat{\mathbf{u}}}{dt} = \mathbf{0} \qquad \cdots \qquad \text{(iii)}$$

But $\hat{\mathbf{u}}$ has a constant length unity. Hence, by theorem 1, we get

$$\hat{\mathbf{u}}\,.\,\frac{d\hat{\mathbf{u}}}{dt} = 0 \qquad \cdots \qquad \text{(iv)}$$

The equations (iii) and (iv) are contradictory unless

$$\frac{d\hat{\mathbf{u}}}{dt}=\mathbf{0} ;$$

that is, the unit vector $\hat{\mathbf{u}}$ is constant which means that $\mathbf{u}$ is parallel to a fixed line.

Note. Readers would do well to compare at this stage Law II and the results of the above two theorems.

5·32. Derivatives of triple products.

As in scalar calculus, derivatives of triple products will be equal to the sum of the quantities got by differentiating a single factor and leaving the others unchanged. We have then the formulae

$$\frac{d}{dt}[\mathbf{a}.(\mathbf{b}\times\mathbf{c})]=\frac{d\mathbf{a}}{dt}.(\mathbf{b}\times\mathbf{c})+\mathbf{a}.\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\mathbf{a}.\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)$$

and $$\frac{d}{dt}[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]=\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})+\mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right).$$

The results can be easily proved by repeated applications of Law V of the previous article. The cyclic order in each term of the first formula should be maintained. The order of the factors in each term of the second formula should also be maintained ; otherwise a change of sign may be necessary.

Examples. V(A)

[**Note :** In the following examples, $\mathbf{r}$ is a function of t ; $\mathbf{a}$ and $\mathbf{b}$ are constant vectors ; r, a, b are the respective lengths of $\mathbf{r}, \mathbf{a}, \mathbf{b}$. Dashes donote differentiations with respect to t].

1. If $\mathbf{r}=t^2\mathbf{i}-t\mathbf{j}+(2t+1)\mathbf{k}$, find at $t=0$ the values of

$$\frac{d\mathbf{r}}{dt};\ \frac{d^2\mathbf{r}}{dt^2};\ \frac{d^3\mathbf{r}}{dt^3};\ \left|\frac{d\mathbf{r}}{dt}\right|;\ \left|\frac{d^2\mathbf{r}}{dt^2}\right|.$$

2. If $\mathbf{r}=5t^2\mathbf{i}+t\mathbf{j}-t^3\mathbf{k}$ and $\mathbf{s}=\cos\theta\mathbf{i}-\sin\theta\mathbf{j}-\theta\mathbf{k}$, find the values of $\frac{d}{dt}(\mathbf{r}.\mathbf{s})$, $\frac{d}{dt}(\mathbf{r}\times\mathbf{s})$, $\frac{d}{dt}(\mathbf{r}.\mathbf{r})$.

3. If $\mathbf{r}_1=(\sin\theta, \cos\theta, \theta)$, $\mathbf{r}_2=(\cos\theta, -\sin\theta, -3)$ and $\mathbf{r}_3=(2, 3, -1)$, find

$$\frac{d}{d\theta}\{\mathbf{r}_1\times(\mathbf{r}_2\times\mathbf{r}_3)\} \text{ at } \theta=0.$$

4. Evaluate the following derivatives :

(i) $\frac{d}{dt}\left\{\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right\}$; (ii) $\frac{d^2}{dt^2}\left\{\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right\}$.

5. Evaluate the following :

(i) $\frac{d}{dt}\left[\mathbf{r}\times(\mathbf{r}'\times\mathbf{r}'')\right]$; (ii) $\frac{d^2}{dt^2}\left[\mathbf{r}\times(\mathbf{r}'\times\mathbf{r}'')\right]$.

6. Find the derivatives (with respect to t) of the following :

(i) $r^2\mathbf{r}+(\mathbf{a}.\mathbf{r})\mathbf{b}$; (ii) $(a\mathbf{r}+r\mathbf{b})^2$; (iii) $\frac{\mathbf{r}}{r}$;

(iv) $r^n\mathbf{r}$; (v) $r^3\mathbf{r}+\mathbf{a}\times\frac{d\mathbf{r}}{dt}$; (vi) $\frac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}.\mathbf{a}}$,

(vii) $\frac{\mathbf{r}}{\mathbf{a}.\mathbf{r}}-\frac{\mathbf{r}\times\mathbf{a}}{r}$; (viii) $\frac{\mathbf{r}}{r^2}+\frac{r\mathbf{b}}{\mathbf{a}.\mathbf{r}}$; (ix) $\mathbf{r}^2+\frac{1}{\mathbf{r}^2}$;

(x) $r^2+\frac{1}{r^2}$; (xi) $\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2$; (xii) $\mathbf{r}\times\mathbf{s}'-\mathbf{r}'\times\mathbf{s}$.

7. Differentiate $\frac{\mathbf{r}+\mathbf{b}}{\mathbf{r}^2+\mathbf{b}^2}$ with respect to t.

8. [If $\mathbf{r}$ be the displacement vector at any time t of a moving point then we shall prove in our subsequent discussions that the velocity-vector $\mathbf{v}=\frac{d\mathbf{r}}{dt}$ and acceleration-vector $=\frac{d^2\mathbf{r}}{dt^2}$ at the instant t].

At an instant t the vector from the origin to a moving point is

$$\mathbf{r}=\mathbf{a}\cos wt+\mathbf{b}\sin wt$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and w is a scalar constant.

(i) Find the velocity $\mathbf{v}$ and show that

$$\mathbf{r}\times\mathbf{v}=\text{a constant vector.}$$

(ii) Show that the acceleration is directed towards the origin and is proportional to the displacement.

(iii) Prove that the velocity $\mathbf{v}$ of the moving point is perpendicular to $\mathbf{r}$.

What is the physical interpretation of this motion ?

9. Verify that

$$\mathbf{r}=\mathbf{a}e^{m_1 t}+\mathbf{b}e^{m_2 t}$$

satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+k\frac{d\mathbf{r}}{dt}+l\mathbf{r}=\mathbf{0} \quad (k,\ l \text{ are scalar constants})$$

where m_1 and m_2 are two unequal roots of the equation

$$m^2+km+l=0$$

and $\mathbf{a}$, $\mathbf{b}$ are arbitrary constant vectors.

Also show that if the two roots of the above quadratic equation are equal (say m_1 and m_1) then

$$\mathbf{r}=(\mathbf{a}+t\mathbf{b})\,e^{m_1 t}$$

will satisfy the differential equation.

10. If $\mathbf{r}=[a\cos t,\ a\sin t,\ at\tan\alpha]$, then evaluate

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^3\mathbf{r}}{dt^3}.$$

Hints and Answers

1. Using Law I of art 5·3 we obtain

$$\mathbf{r}'=(2t,\ -1,\ 2)\ ; \qquad \text{at } t=0,\ \mathbf{r}'=(0,\ -1,\ 2).$$

$$\mathbf{r}''=(2,\ 0,\ 0)\ ; \qquad \text{at } t=0,\ \mathbf{r}''=(2,\ 0,\ 0).$$

$$\mathbf{r}'''=(0,\ 0,\ 0)\ ; \qquad \text{at } t=0,\ \mathbf{r}'''=(0,\ 0,\ 0).$$

Hence $|\mathbf{r}'|=\sqrt{0+1+4}=\sqrt{5}\ ;\ |\mathbf{r}''|=\sqrt{4+0+0}=2.$

2. (i) $\dfrac{d}{dt}(\mathbf{r}.\mathbf{s})=\dfrac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}.\dfrac{d\mathbf{s}}{dt}$

$=(10t,1,\ -3t^2).(\sin t,\ -\cos t,\ 0)+(5t^2,\ t,\ -t^3).(\cos t,\ \sin t,\ 0)$

$=10t\sin t-\cos t+5t^2\cos t+t\sin t=(5t^2-1)\cos t+11\,t\sin t.$

Alternatively. Obtain $\mathbf{r}.\mathbf{s}=5t^2 \sin t-t\cos t$. Then

$$\frac{d}{dt}(\mathbf{r}.\mathbf{s})=\frac{d}{dt}(5t^2 \sin t-t\cos t)=(5t^2-1)\cos t+11t\sin t.$$

(ii) $\frac{d}{dt}(\mathbf{r}\times\mathbf{s})=(t^3 \sin t-3t^2\cos t)\,\mathbf{i}-(t^3\cos t+3t^2\sin t)\,\mathbf{j}$
$+(5t^2\sin t-\sin t-11t\cos t)\,\mathbf{k}.$

(iii) $\frac{d}{dt}(\mathbf{r}.\mathbf{r})=2\mathbf{r}.\frac{d\mathbf{r}}{dt}=2(5t^2\mathbf{i}+t\mathbf{j}-t^3\mathbf{k}).(10t\mathbf{i}+\mathbf{j}-3t^2\mathbf{k})=100t^3+2t+6t^5.$

3. $7\mathbf{i}+6\mathbf{j}-6\mathbf{k}$.

4. (i) We use the formula of art. 5·32.

$$\frac{d}{dt}\left\{\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right\}=\mathbf{r}'.\mathbf{r}'\times\mathbf{r}''+\mathbf{r}.\mathbf{r}''\times\mathbf{r}''+\mathbf{r}.\mathbf{r}'\times\mathbf{r}'''.$$

Since the scalar triple product vanishes if two of the three vectors in the box are same, the first two terms on the right vanish. Hence the required product is the box $[\,\mathbf{r}\,\mathbf{r}'\mathbf{r}'''\,]$.

(ii) Taking derivative of $[\,\mathbf{r}\,\mathbf{r}'\mathbf{r}'''\,]$ it can be shown that this *second* derivative will be $[\,\mathbf{r}\mathbf{r}''\mathbf{r}'''\,]+[\,\mathbf{r}\mathbf{r}'\mathbf{r}''''\,]$.

5. (i) $\mathbf{r}'\times(\mathbf{r}'\times\mathbf{r}'')+\mathbf{r}\times(\mathbf{r}'\times\mathbf{r}''')$;

(ii) $\mathbf{r}''\times(\mathbf{r}'\times\mathbf{r}'')+2\mathbf{r}'\times(\mathbf{r}'\times\mathbf{r}''')+\mathbf{r}\times(\mathbf{r}''\times\mathbf{r}''')+\mathbf{r}\times(\mathbf{r}'\times\mathbf{r}'''')$.

6. (i) $2rr'\mathbf{r}+r^2\mathbf{r}'+(\mathbf{a}.\mathbf{r}')\,\mathbf{b}$; (ii) $2(a\mathbf{r}+r\mathbf{b}).(a\mathbf{r}'+r'\mathbf{b})$;

(iii) $\frac{1}{r^2}\left(r\frac{d\mathbf{r}}{dt}-\mathbf{r}\frac{dr}{dt}\right)$; (iv) $nr^{n-1}\frac{dr}{dt}\mathbf{r}+r^n\frac{d\mathbf{r}}{dt}$;

(v) $3r^2\frac{dr}{dt}\mathbf{r}+r^3\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}$; (vi) $\frac{\mathbf{r}'\times\mathbf{a}}{\mathbf{r}.\mathbf{a}}-\frac{(\mathbf{r}.\mathbf{a})(\mathbf{r}\times\mathbf{a})}{(\mathbf{r}.\mathbf{a})^2}$.

(vii) $\frac{(\mathbf{a}.\mathbf{r})\mathbf{r}'-(\mathbf{a}.\mathbf{r}')\mathbf{r}}{(\mathbf{a}.\mathbf{r})^2}-\frac{r(\mathbf{r}'\times\mathbf{a})-(\mathbf{r}\times\mathbf{a})\,r'}{r^2}$;

(viii) $\frac{1}{r^2}\mathbf{r}'-\frac{2r'}{r^3}\mathbf{r}+\frac{r'}{\mathbf{a}.\mathbf{r}}\mathbf{b}-\frac{r\,(\mathbf{a}.\mathbf{r}')\,\mathbf{b}}{(\mathbf{a}.\mathbf{r})^2}$;

(ix) and (x) are identical problems since $\mathbf{r}^2=r^2$. The required derivative in each case will be $2rr'-2r'/r^3$;

(xi) $m\mathbf{r}'.\mathbf{r}''$; (xii) $\mathbf{r}\times\mathbf{s}''-\mathbf{r}''\times\mathbf{s}$.

7. $\frac{(\mathbf{r}^2+\mathbf{b}^2)\,\mathbf{r}'-2rr'(\mathbf{r}+\mathbf{b})}{(\mathbf{r}^2+\mathbf{b}^2)^2}$.

8. $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=w\,(-\mathbf{a}\sin wt+\mathbf{b}\cos wt)$;

$$\frac{d^2\mathbf{r}}{dt^2}=-w^2\,(\mathbf{a}\cos wt+\mathbf{b}\sin wt).$$

(i) $\mathbf{r}\times\mathbf{v}=(\mathbf{a}\cos wt+\mathbf{b}\sin wt)\times w\,(-\mathbf{a}\sin wt+\mathbf{b}\cos wt)$

$$=w\,(\cos^2 wt+\sin^2 wt)\,\mathbf{a}\times\mathbf{b}=w\mathbf{a}\times\mathbf{b}, \text{ a constant vector.}$$

(ii) Clearly, $\dfrac{d^2\mathbf{r}}{dt^2}=-w\mathbf{r}$. Then the acceleration is opposite to the direction of $\mathbf{r}$ *i.e.*, it is directed towards the origin. Its magnitude is proportional to $|\,\mathbf{r}\,|$ which is the distance from the origin.

(iii) Verify that $\mathbf{r}\,.\,\mathbf{v}=0$; hence etc.

Physical interpretation. The motion is that of a particle moving on the circumference of a circle with constant angular speed w. The force, directed towards the centre of the circle, is the *centrepetal force.*

10. $\mathbf{r}'=(-a\sin t,\ a\cos t,\ a\tan\alpha)$; $\mathbf{r}''=(-a\cos t,\ -a\sin t,\ 0)$.

$\therefore\ \mathbf{r}'\times\mathbf{r}''=(-a^2\sin t\tan\alpha,\ -a^2\cos t\tan\alpha,\ a^2)$;

$|\,\mathbf{r}'\times\mathbf{r}''\,|=a^2\sec\alpha.$

Second part, $\mathbf{r}'''=(a\sin t,\ -a\cos t,\ 0)$. Now obtain $\mathbf{r}'.\,\mathbf{r}''\times\mathbf{r}'''$ in the form of the determinant

$$\begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

Now evaluate the determinant.

5·4. Differentials : Indefinite Integration with respect to a scalar variable.

Differentials.

We introduce the concept of differentials exactly in the same way as we do in scalar calculus (See for recapitulation, *Maity and Ghosh, Differential Calculus*, art. 7·6 and 9·11).

Thus when $\mathbf{F}'(t)$ exists we call the vector $\mathbf{F}'(t)dt$ as the differential of $\mathbf{F}(t)$ and denote it by $d\mathbf{F}$; dt is same as the increment of the scalar (independent) variable t. We now write,

$$d\mathbf{F}=\mathbf{F}'(t)\,dt.$$

Now if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ then $d\mathbf{F} = dF_1\mathbf{i} + dF_2\mathbf{j} + dF_3\mathbf{k}$, and in particular, if $\mathbf{c}$ be a constant vector then $d\mathbf{c} = \mathbf{0}$.

Also $\quad d(\mathbf{F}.\mathbf{f}) = \mathbf{F}.d\mathbf{f} + d\mathbf{F}.\mathbf{f}$; $d(\mathbf{F} \times \mathbf{f}) = \mathbf{F} \times d\mathbf{f} + d\mathbf{F} \times \mathbf{f}$.

Indefinite Integration.

If $\mathbf{F}(t)$ be the derivative of the function $\mathbf{f}(t)$ *i. e.*, if $d\mathbf{f}(t) = \mathbf{F}(t)dt$ then we say that $\mathbf{f}(t)$ is an *indefinite integral* or simply *integral* of $\mathbf{F}(t)$ with respect to t and write

$$\int \mathbf{F}(t)dt = \mathbf{f}(t) \qquad \cdots \quad (1)$$

The *process of integration* is thus to search for the vector function $\mathbf{f}(t)$ whose derivative with respect to t is the given vector function $\mathbf{F}(t)$. Thus this process is reverse of differentiation.

Note that this definition is same as the corresponding definition in Scalar Calculus (see *Maity and Bagchi, Integral Calculus*, Part I, art. 1·2).

We call $\mathbf{f}(t)$ the *integral* and $\mathbf{F}(t)$ the *integrand* so that we may remember that the derivative of the integral is equal to the integrand.

Theorem. *If* $\mathbf{f}(t)$ *be an indefinite integral of* $\mathbf{F}(t)$ *then* $\mathbf{f}(t) + \mathbf{c}$, *where* $\mathbf{c}$ *is a constant vector is also the integral of* $\mathbf{F}(t)$.

Proof. First method. The given condition gives

$$d\{\mathbf{f}(t)\} = \mathbf{F}(t)\,dt.$$

But it is also true that $d\{\mathbf{f}(t) + \mathbf{c}\} = d\{\mathbf{f}(t)\} + d\mathbf{c} = \mathbf{F}(t)\,dt$

($\because$ $\mathbf{c}$ is a constant vector $d\mathbf{c} = \mathbf{0}$).

Hence, according to definition, $\mathbf{f}(t) + \mathbf{c}$ is also an integral of $\mathbf{F}(t)$.

Second method. Suppose $\mathbf{f}_1(t)$ is another integral of $\mathbf{F}(t)$ besides $\mathbf{f}(t)$, then we have

$$d\{\mathbf{f}_1(t) - \mathbf{f}(t)\} = d\{\mathbf{f}_1(t)\} - d\{\mathbf{f}(t)\} = \mathbf{F}(t)\,dt - \mathbf{F}(t)\,dt = \mathbf{0}.$$

This implies that $\mathbf{f}_1(t)-\mathbf{f}(t)$ is a constent vector $\mathbf{c}$ (say)

$$\therefore \quad \mathbf{f}_1(t)=\mathbf{f}(t)+\mathbf{c}.$$

Note : The word *indefinite integral* has been suggested because of the arbitrariness of the constant vector $\mathbf{c}$. This constant is known as the *constant of integration.* In a practical problem we will be given some such condition which will give a definite value for $\mathbf{c}$.

5·41. Important formulæ.

We append below a list of formulæ which the reader will do well to remember. We denote the two derivable vector functions $\mathbf{r}(t)$ and $\mathbf{s}(t)$ by $\mathbf{r}$ and $\mathbf{s}$ respectively.

I. $$\int\left(\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}+\mathbf{s}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}.\mathbf{s}+c$$

For, $\frac{d}{dt}(\mathbf{r}.\mathbf{s})=\mathbf{r}\,.\,\frac{d\mathbf{s}}{dt}+\mathbf{s}\,.\,\frac{d\mathbf{r}}{dt}$ and hence $d(\mathbf{r}.\mathbf{s})=\left(\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}+\mathbf{s}\cdot\frac{d\mathbf{r}}{dt}\right)dt.$

II. $$\int 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\,dt=\mathbf{r}.\mathbf{r}+c=\mathbf{r}^2+c$$

(Put $\mathbf{r}=\mathbf{s}$ in I)

III. $$\int 2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\,dt=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+c=\left(\frac{d\mathbf{r}}{dt}\right)^2+c$$

(Replace $\mathbf{r}$ by $\frac{d\mathbf{r}}{dt}$ in II)

IV. $$\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$$

For, $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$ and hence $d\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt$

V. $$\int\left(\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{dr}{dt}\,\frac{\mathbf{r}}{r^2}\right)dt=\frac{\mathbf{r}}{r}+\mathbf{c}$$

For, $\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)=\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{dr}{dt}\,\frac{\mathbf{r}}{r^2}$ and hence $d\left(\frac{\mathbf{r}}{r}\right)=\left(\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{dr}{dt}\,\frac{\mathbf{r}}{r^2}\right)dt.$

VI. $\int \mathbf{a} \times \frac{d\mathbf{r}}{dt}\, dt = \mathbf{a} \times \mathbf{r} + \mathbf{c}$

For, $d\,(\mathbf{a} \times \mathbf{r}) = \left(\mathbf{a} \times \frac{d\mathbf{r}}{dt}\right) dt$, **a** being a given constant vector.

Important Note : The constant of integration is a scalar or a vector according as the integrand is a scalar or a vector function. Verify the truth of this statement in each of the above cases.

5·5. Definite Integral : Integration as limit of a sum.

Let $\mathbf{f}(t)$ be a given vector function of the scalar variable t in the interval (a, b). To make the matter simple, we suppose that $\mathbf{f}(t)$ is *finite* and *continuous* for all values of t ranging from a to b. Let the interval $b-a$ be divided into a finite number of sub-intervals (say n) which correspond to *increments*

$$\triangle t_1, \triangle t_2, \ldots, \triangle t_n$$

of the variable t. We take *some value* t_1 for t in the first sub-interval, t_2 *some value* for t in the second sub-interval and so on ; the corresponding values of the function are respectively

$$\mathbf{f}(t_1), \mathbf{f}(t_2) \ldots, \mathbf{f}(t_n).$$

Now we form the sum

$$\mathbf{S} = \triangle t_1\, \mathbf{f}(t_1) + \triangle t_2\, \mathbf{f}(t_2) + \cdots\cdots + \triangle t_n\, \mathbf{f}(t_n).$$

Next let the number of sub-intervals increase indefinitely so that the greatest of the increments $\triangle t$ tends to zero. Then it can be shown that the sum-vector **S** tends to a unique finite limit, independent of the mode of sub-division of the range $b-a$ and also of the choice of the values $t_1, t_2, \ldots t_n$ in these sub-intervals. This limiting value of **S** can be shown to be equal to the difference of the values of the indefinite integral $\mathbf{F}(t)$ of the function $\mathbf{f}(t)$ for $t=b$ and $t=a$; that is,

$$\lim_{n \to \infty} \mathbf{S} = \mathbf{F}(b) - \mathbf{F}(a), \text{ where } \mathbf{F}(t) = \int \mathbf{f}(t)\, dt$$

This limiting value is called the *definite integral* of the function $\mathbf{f}(t)$ between the limits a and b and is denoted by

$$\int_a^b \mathbf{f}(t)\,dt$$

Thus, $$\int_a^b \mathbf{f}(t)\,dt = \lim_{n\to\infty} \sum_{r=1}^{n} \Delta t_r\, \mathbf{f}(t_r) = \mathbf{F}(b) - \mathbf{F}(a)$$

where $\mathbf{F}(t) = \int \mathbf{f}(t)\,dt.$

5·51. Examples on Integration as Summation.

I. Vector sum as integration.

A curve consists of an infinite number of vectors whose lengths are indefinitely small and directions are along the tangent to the curve (Fig. 5.2). If $d\mathbf{a}$ represents any one of these small vectors then by adding them all (using vector law of Addition) we get the resultant vector $\overrightarrow{AB}$. The sum of these infinite number of small vectors :

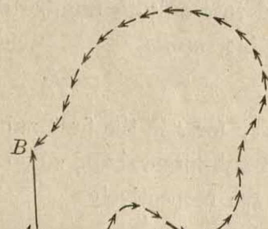


Fig. 5·2.
Vector sum as integration

$$= \lim d\mathbf{a} = \overrightarrow{AB} = \int_A^B d\mathbf{a}.$$

If the curve is a closed one, whether a plane curve or not, then A and B coincide and $\overrightarrow{AB} = \mathbf{0}$. *i.e.*, $\int d\mathbf{a}$ around a closed path is zero.

II. Total displacement as an integration.

Consider the displacement of a moving point P. Suppose at time t the velocity of P is given by vector $\mathbf{v}$. As t varies, the velocity $\mathbf{v}$ also changes *i.e.*, $\mathbf{v}$ is a function of t. Suppose we are to find the total displacement of P as t varies from t_0 to t_1. We divide the range $t_1 - t_0$ into a number of sub-intervals. We consider a *typical* sub-interval whose length is Δt. Displace-

ment in this sub-interval is $\mathbf{v}\triangle t$ where $\mathbf{v}$ is the velocity, supposed to be constant in that small sub-interval. The total displacement during the interval from t_0 to t_1 is, then,

$$\lim_{\Delta t\to 0} \Sigma \mathbf{v}\triangle t = \int_{t_0}^{t_1} \mathbf{v}dt.$$

III. Total increment in velocity as an integration.

In the last example suppose the moving point P has a variable acceleration **a**, which is, of course, a function of t. In order to find the total increment in velocity during the interval t_0 to t_1 we, as before, divide the interval into a number of sub-intervals. During one of these, whose duration is $\triangle t$, the increment in velocity is $\mathbf{a}\,\triangle t$ where **a** is the constant acceleration for that sub-interval. Hence total increment in velocity is given by

$$\lim_{\Delta t\to 0} \Sigma\, \mathbf{a}\triangle t = \int_{t_0}^{t_1} \mathbf{a}\, dt.$$

5·6. Linear vector differential equation.

Linear differential equations containing one or more variable vectors and their derivatives with respect to scalars can be solved like scalar differential equations, the main difference being that the *constant of integration are vectors* (*not scalars*). The following illustrations will make the point clear.

I. Motion under a constant acceleration.

Consider the motion of a particle under a constant acceleration

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \qquad (\mathbf{a} \text{ is a constant vector}).$$

By direct integration, we obtain

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \mathbf{a}t + \mathbf{v}_0 \;\; (\mathbf{v}_0 \text{ is an arbitrary constant vector}) \;\cdots \text{(i)}$$

We can determine the value $\mathbf{v}_0$ under some known initial condition ; suppose we know the value of $\frac{d\mathbf{r}}{dt}$ when $t=0$.

A second integration gives

$$\mathbf{r} = \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0 \qquad \cdots \quad \text{(ii)}$$

where $\mathbf{r}_0$ is another arbitrary constant vector and its value is that of $\mathbf{r}$ when $t=0$. The last two equations (i) and (ii) give the velocity and displacement at any time t.

II. Harmonic Motion.

As another example, suppose we are to integrate

$$\frac{d^2\mathbf{r}}{dt^2} + w^2\mathbf{r} = \mathbf{0} \qquad \cdots \quad \text{(iii)}$$

which is the equation of a central acceleration, *i.e.*, a motion which is always directed towards or away from the centre ; the law of force as a function of the distance is, in this case, such that the magnitude of acceleration is proportional to the distance from the centre.

We can not write down the integral of the second member, since $\mathbf{r}$ is not a known function of t. But on forming the scalar product of each side with $2\frac{d\mathbf{r}}{dt}$, we obtain

$$2\frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} \cdot w^2\mathbf{r} = \mathbf{0}$$

$$i.e.,\ \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2 + 2w^2\,\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

On integration,

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 + w^2\mathbf{r}^2 = c$$

where c is a scalar constant. (Since each of the integrand on the left side is a scalar.) We now require to integrate

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = c - w^2\mathbf{r}^2 \qquad \cdots \quad \text{(iv)}$$

This will present some difficulty. So we proceed in the following indirect way :

We know that the two solutions of the scalar equation

$$\frac{d^2r}{dt^2}+w^2r=0$$

are $r=a\cos wt$ and $r=b\sin wt$ and that the complete solution is the sum of these two. If we replace the arbitrary scalar constants a and b by the arbitrary constant vectors **a** and **b** obtaining

$$\mathbf{r}=\mathbf{a}\cos wt+\mathbf{b}\sin wt \qquad \cdots \quad (\text{v})$$

it will be easily seen, by differentiation that this equation (v) is the complete solution of the vector differential equation (iii).

By an extension of this process, we now state the rule for solution of linear differential equations to any order with constant coefficients :

Find the solutions, assuming the vector variable to be a scalar variable, multiply each of these solutions by an arbitrary vector and then add. The result is the complete solution of the vector differential equation.

III. Simultaneous Equations.

Suppose that the vectors **X**, **Y** are functions of t which satisfy the equations :

$$\frac{d\mathbf{X}}{dt}=k\mathbf{Y}\ ;\ \frac{d\mathbf{Y}}{dt}=-k\mathbf{X}$$

where k is a scalar constant.

Differentiating the first equation and replacing $\frac{d\mathbf{Y}}{dt}$ from the second, we obtain

$$\frac{d^2\mathbf{X}}{dt^2}+k^2\mathbf{X}=\mathbf{0}.$$

The solution of this equation is, as obtained before,

$$\mathbf{X}=\mathbf{A}\sin kt+\mathbf{B}\cos kt$$

where **A** and **B** are arbitrary constant vectors. Substituting this value in the first equation, we get

$$\mathbf{Y}=\mathbf{A}\cos kt-\mathbf{B}\sin kt.$$

Examples. V(B)

1. If $\mathbf{r}(t)=(3t^2-1)\,\mathbf{i}+(2-6t)\,\mathbf{j}-4t\mathbf{k}$, show that

$$\int \mathbf{r}(t)\,dt=(t^3-\tfrac{1}{2}t^2)\,\mathbf{i}+(2t-3t^2)\,\mathbf{j}-2t^2\mathbf{k}+\mathbf{c}\,;$$

and $\displaystyle\int_1^2 \mathbf{r}(t)\,dt=\frac{11}{2}\,\mathbf{i}-7\mathbf{j}-6\mathbf{k}.$

2. Similar problem with $\mathbf{r}(t)=(t-t^2)\,\mathbf{i}+2t^3\mathbf{j}-3\mathbf{k}.$

3. If $\mathbf{r}(t)=(t,\,-t^2,\,t-1)$, $\mathbf{s}(t)=(2t^2,\,0,\,6t)$, show that

$$\int_0^2 \mathbf{r}\cdot\mathbf{s}\,dt=12 \qquad \text{and} \int_0^2 \mathbf{r}\times\mathbf{s}\,dt=\left(-24,\,-\frac{40}{3},\,\frac{64}{5}\right).$$

4. If $\mathbf{r}_1(t)=t\mathbf{i}-3\mathbf{j}+2t\mathbf{k}$, $\mathbf{r}_2(t)=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, $\mathbf{r}_3(t)=3\mathbf{i}+t\mathbf{j}-\mathbf{k}$,

then $\displaystyle\int_1^2 \mathbf{r}_1\cdot\mathbf{r}_2\times\mathbf{r}_3\,dt=0\,;$

$$\int_1^2 \mathbf{r}_1\times(\mathbf{r}_2\times\mathbf{r}_3)\,dt=-\tfrac{1}{2}\,87\mathbf{i}-\tfrac{1}{3}\,44\mathbf{j}+\tfrac{1}{2}\,15\mathbf{k}.$$

5. (i) Prove that

$$\int_1^2 \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\,dt=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k},$$

where $\mathbf{r}(t)=5t^2\mathbf{i}+t\mathbf{j}-t^3\mathbf{k}.$

(ii) With the same vector function $\mathbf{r}(t)$ evaluate

$$\int_1^2\left(\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{dr}{dt}\,\frac{\mathbf{r}}{r^2}\right)dt.$$

(iii) Given that

$$\begin{aligned}\mathbf{r}(t)&=2\mathbf{i}-\mathbf{j}+2\mathbf{k} &&\text{when } t=2\\ &=4\mathbf{i}-2\mathbf{j}+3\mathbf{k} &&\text{when } t=3\end{aligned}$$

Show that

$$\int_2^3 \mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\,dt=10.$$

6. (i) The acceleration at any instant t of a moving particle is given by

$$\frac{d^2\mathbf{r}}{dt^2}=12\cos 2t\,\mathbf{i}-8\sin 2t\,\mathbf{j}+16t\mathbf{k}$$

find the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ at any time t, and if at time $t=0$, it is known that $\mathbf{v}=\mathbf{0}$ and $\mathbf{r}=\mathbf{0}$.

(ii) Find $\mathbf{r}$ at any instant t if

$$\frac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4\sin t\mathbf{k};$$

(Given : $\mathbf{r}=2\mathbf{i}+\mathbf{j}$ and $\dfrac{d\mathbf{r}}{dt}=-\mathbf{i}-3\mathbf{k}$ at $t=0$).

7. Solve for $\mathbf{r}$:

(i) $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$, where $\mathbf{a}$ is a constant vector (given).

Obtain also the particular solution if it be given that

when $t=0$, $\mathbf{r}=\mathbf{0}$ and $\dfrac{d\mathbf{r}}{dt}=\mathbf{u}$.

(ii) $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}t+\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are known constant vectors.

(Given that when $t=0$, $\mathbf{r}$ as well as $\dfrac{d\mathbf{r}}{dt}$ are both equal to $\mathbf{0}$).

8. Solve the vector equations for $\mathbf{x}$

(i) $p\mathbf{x}+(\mathbf{x}\cdot\mathbf{b})\,\mathbf{a}=\mathbf{c}$ $\quad(p\neq 0)$.

(ii) $\mathbf{x}\times\mathbf{a}=\mathbf{b}$ $\quad(\mathbf{a}\cdot\mathbf{b}=0)$.

and, if necessary, use (ii) to solve the differential equation

$$\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{b}\qquad(\mathbf{a}\cdot\mathbf{b}=0)$$

($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are given constant vectors).

9. Show that the general solution of

$$\frac{d^2\mathbf{r}}{dt^2}+2a\frac{d\mathbf{r}}{dt}+b\mathbf{r}=\mathbf{0},$$

where a and b are scalar constants, is

$$\begin{aligned}\mathbf{r} &= e^{-at}\left(\mathbf{c}_1\, e^{\sqrt{a^2-b}\,t} + \mathbf{c}_2\, e^{-\sqrt{a^2-b}\,t}\right); && \text{if } a^2-b>0\\ &= e^{-at}\left(\mathbf{c}_1 \sin\sqrt{a^2-b}\,t + \mathbf{c}_2 \cos\sqrt{a^2-b}\,t\right), && \text{if } a^2-b<0\\ &= e^{-at}\left(\mathbf{c}_1+\mathbf{c}_2 t\right), && \text{if } a^2-b=0.\end{aligned}$$

$\mathbf{c}_1$, $\mathbf{c}_2$ being arbitrary constant vectors.

Hence solve the following three equations

$$\frac{d^2\mathbf{r}}{dt^2} - 7\frac{d\mathbf{r}}{dt} + 12 = \mathbf{0}\ ; \quad \frac{d^2\mathbf{r}}{dt^2} - 6\frac{d\mathbf{r}}{dt} + 9 = \mathbf{0}\ ; \quad \frac{d^2\mathbf{r}}{dt^2} + 4\mathbf{r} = \mathbf{0}.$$

Hints and Answers

1. $\int \mathbf{r}\, dt = \mathbf{i} \int (3t^2-1)\, dt + \mathbf{j} \int (2-6t)\, dt - 4\mathbf{k} \int t\, dt = \mathbf{F}(t) + \mathbf{c}$

= required result.

$$\int_1^2 \mathbf{r}\, dt = \mathbf{F}(2) - \mathbf{F}(1).$$

2. $(\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3)\,\mathbf{i} + \tfrac{1}{2}t^4\mathbf{j} - 3t\mathbf{k} + \mathbf{c}\ ;\ -\tfrac{5}{6}\mathbf{i} + \tfrac{15}{2}\mathbf{j} - 3\mathbf{k}.$

5. (i), (ii), (iii). See the important formulæ of art. 5·41.

6. (i) Integrating,

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt}\\ &= 12\mathbf{i} \int \cos 2t\, dt - 8\mathbf{j} \int \sin 2t\, dt + \mathbf{k} \int 16t\, dt\\ &= 6 \sin 2t\, \mathbf{i} + 4 \cos 2t\, \mathbf{j} + 8t^2\, \mathbf{k} + \mathbf{c}_1.\end{aligned}$$

Initially $\mathbf{v}=\mathbf{0}$ when $t=0$. This gives $\mathbf{c}_1 = -4\mathbf{j}$.

Thus, $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 6 \sin 2t\, \mathbf{i} + 4 \cos 2t\, \mathbf{j} + 8t^2\mathbf{k} - 4\mathbf{j}$.

Integrating again and using the initial condition $\mathbf{r}=\mathbf{0}$ when $t=0$ we easily obtain,

$$\mathbf{r} = (3 - 3 \cos 2t)\, \mathbf{i} + (2 \sin 2t - 4t)\, \mathbf{j} + \tfrac{1}{3}8t^3\, \mathbf{k}.$$

(ii) Proceed in a similar manner with

$$\mathbf{r} = (t^3 - t + 2)\, \mathbf{i} + (1 - 2t^4)\, \mathbf{j} + (t - 4 \sin t)\, \mathbf{k}.$$

7. (i) General solution is $\mathbf{r} = \tfrac{1}{2}\,\mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors of which the first one is given and the other two arbitrary. Particular solution is $\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\,\mathbf{a}t^2$.

(ii) $\mathbf{r} = \tfrac{1}{6}\,\mathbf{a}t^3 + \tfrac{1}{2}\,\mathbf{b}t^2$.

8. (i) Form the dot product of both members with $\mathbf{b}$ and then eliminate $\mathbf{x}\,.\,\mathbf{b}$. This will give

$$\mathbf{x}=-\frac{\mathbf{b}\,.\,\mathbf{c}}{p\,(p+\mathbf{a}\,.\,\mathbf{b})}\,\mathbf{a}+\frac{\mathbf{c}}{p}\,;$$

this is the required solution provided $p+\mathbf{a}\,.\,\mathbf{b}\neq 0$. If however $p+\mathbf{a}\,.\,\mathbf{b}=0$, the equation is satisfied by $\mathbf{x}=\mathbf{c}/p-\lambda\mathbf{a}$ for any value of λ.

Alternatively, the form of the given equation suggests that $\mathbf{x}$ should be expressible as a linear combination of $\mathbf{a}$ and $\mathbf{c}$ and hence may be assumed to be of the form

$$\mathbf{x}=\frac{\mathbf{c}}{p}-\lambda\mathbf{a},$$

where λ can be obtained by substituting this value of $\mathbf{x}$ in the given equation. The value of λ can be easily found to be $-\dfrac{\mathbf{b}\,.\,\mathbf{c}}{p\,(p+\mathbf{a}\,.\,\mathbf{b})}$; hence etc.

(ii) Form the cross product of each member by $\mathbf{a}$. Then

$$\mathbf{a}^2\mathbf{x}-(\mathbf{a}\,.\,\mathbf{x})\,\mathbf{a}=\mathbf{a}\times\mathbf{b}$$

which is of the same form as (i) above and the general solution can be found to be

$$\mathbf{x}=\lambda\mathbf{a}+\frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}^2},$$

with λ as parameter.

Third part. Solve for $\mathbf{x}=\dfrac{d^2\mathbf{r}}{dt^2}$ as in (ii) *i.e.*, find $\mathbf{x}$ from

$$\mathbf{a}\times\mathbf{x}=\mathbf{b}\qquad(\mathbf{a}\,.\,\mathbf{b}=0)$$

and obtain $\mathbf{x}=\dfrac{d^2\mathbf{r}}{dt^2}=\lambda\mathbf{a}+\dfrac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}^2}$.

Now integrate twice and obtain the solution

$$\mathbf{r}=\lambda\mathbf{a}+\tfrac{1}{2}t^2\,\frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}^2}+t\mathbf{c}+\mathbf{d}$$

where λ is an arbitrary scalar, and $\mathbf{c}$ and $\mathbf{d}$ are arbitrary constant vectors.

9. Use the rule given in art. 5·6 under No. II. The solutions of last three equations are :

$$\mathbf{r}=\mathbf{c}_1e^{3t}+\mathbf{c}_2e^{4t}\,;\;\;\mathbf{r}=(\mathbf{c}_1+\mathbf{c}_2t)\,e^{3t}\,;\;\;\mathbf{r}=\mathbf{c}_1\cos 2t+\mathbf{c}_2\sin 2t.$$

6

Elements of Differential Geometry

6·1. Introduction.

In this chapter we propose to discuss a few elementary concepts of Differential Geometry of curves. Within the limits of a single chapter it is impossible to examine fully the topics that arise in this connection. We, therefore, lay no claim to the details of the kind that is generally found in a treatise on Differential Geometry. Our main object is to obtain some interesting and useful results, as well as a clearer insight into the Calculus of vectors, by the following applications to Geometry.

6·2. Curves in space.

Definition. A curve is an aggregate of points whose coördinates are functions of a single variable. Thus the equations

$$x=x(t), \qquad y=y(t), \qquad z=z(t) \qquad \cdots \quad (1)$$

represent a curve in space. The variable t is called a *parameter*, and each value of the parameter within a certain range (say, $a \leqslant t \leqslant b$) corresponds to a definite point $P:(x, y, z)$ of the curve.

Examples.

(i) The straight line through the point (a_1, a_2, a_3) with direction ratios r_1, r_2, r_3 has the parametric representation :

$$x=r_1 t+a_1, \quad y=r_2 t+a_2, \quad z=r_3 t+a_3.$$

Note that there is just one point P of the line corresponding to a given value of t, and just one value of t corresponding to a given point P. We

express this fact by stating that there is a one-to-one correspondence between the values of the parameter t and the points of the line.

(ii) The circle in (x, y)-plane whose centre is at the origin and whose radius is a has the parametric equations :

$$x = a\cos t,\ \ y = a\sin t,\ \ z = 0.$$

Note that these equations as such do not establish a one-to-one correspondence between the values of t and the points of the circle which they represent. There is a single point P of the circle corresponding to a given value of t. But there are infinitely many values of t which determine a given point P of the circle ; if t_1 be one of them the others are $t_1 + 2k\pi$ $(k = 1, 2, \ldots, -1, -2, \ldots)$. However, if the parameter t is restricted to lie within the interval $0 \leq t < 2\pi$ or some such suitable interval, t is uniquely determined when P is given.

(iii) The parametric equations

$$x = a\cos t,\ \ y = a\sin t,\ \ z = bt\ (b \neq 0),$$

represent a curve on the surface of a circular cylinder (obtained by drawing the lines through the points of the circle (ii) parallel to the z-axis) and cutting the generators at a constant angle (See Fig. 6.1). This curve is known as a *Circular helix*.

Mechanically this curve is described by a point P which is subjected to a constant rotation together with a translation in the direction of the axis of rotation (*e.g.*, the curve traced by the pen of a *Chronograph*, an astronomical instrument for accurately observing the instant when a heavenly body crosses the meridian).

This curve is an example of a space curve which is *twisted*, that is, which does not lie in a plane,

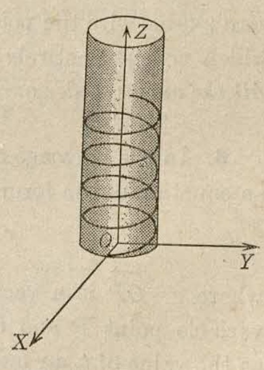


Fig. 6.1. Circular helix

The parametric equations of the circular helix establishes a one-to-one correspondence between the points of the curve and the values of t.

(iv) Another example of a twisted curve is the *twisted cubic* :

$$x = at,\ \ y = bt^2,\ \ z = ct^3\ \ (abc \neq 0).$$

The name *cubic* is suggested by the fact that it is cut by a plane, in general, in three points, real or imaginary.

Remarks on space curves.

1. We refer to the curve given by (1) above. We shall always assume that $x(t)$, $y(t)$ and $z(t)$ are *real, single-valued functions* of t which are defined in a certain interval of t, *say in* $a \leqslant t \leqslant b$, and are also *continuous* in this interval so that the curves that we consider are *continuous* in the defined interval. In order that a curve may not degenerate into a point we exclude the case in which all the three functions are constants. We further restrict the interval (a, b) so that there is just one value of t corresponding to each point of the curve. Then the equations (1) set up a one-to-one correspondence between the points of a curve and the values of t in the interval (a, b).

2. A parameter value t corresponds to an *ordinary point* of the curve when the three derivatives $dx/dt = x'(t)$, $dy/dt = y'(t)$, $dz/dt = z'(t)$ exist, and are continuous at t and at least one is not zero ; otherwise the point is a *singular point*. An arc of a curve which consists entirely of ordinary points is said to be *smooth*. *All the curves under our discussions will be smooth.*

3. In the language of vectors, a curve can be represented by an equation of the form

$$\mathbf{r} = \mathbf{f}(t) \qquad \cdots \quad (2)$$

where $\mathbf{r} = \overrightarrow{OP}$ is a vector drawn from a fixed origin O to the variable point P and the terminus of P can be located as soon as the value of t, an independent scalar variable, is given. As t varies continuously, the terminal point P of $\mathbf{r}$ describes a curve in space (See Fig. 6.2).

Choosing the three fixed directions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, mutually perpendicular, we may express equation (2) analytically as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

which is equivalent to three scalar equations given by (1).

Frequently the parameter that we take for the curve (1) is the length of the arc measured from a fixed point A on the curve upto the variable point P ; we denote this parameter by s to avoid the risk of confusion and write

$$\left.\begin{array}{c}\mathbf{r}=\mathbf{f}(s)\\ \text{or}\quad x=x(s),\, y=y(s),\, z=z(s)\end{array}\right\} \quad \cdots \quad (3)$$

as the equation of the curve in space ; the parameter t will be used in other cases.

6·3. Tangent to a curve at a point.

DEFINITION. The tangent line PT at a point P of a curve is the limiting position of the secant PQ joining P to a neighbouring point Q, when Q approaches P along the curve ; See Fig. 6.2.

1. Tangent line at a point $P(\mathbf{r})$ of the curve $\mathbf{r}=\mathbf{f}(t)$.

We *first* show that the vector $\mathbf{f}'(t)$ is parallel to the tangent line at P (whose position vector is $\mathbf{r}$) on the curve $\mathbf{r}=\mathbf{f}(t)$.

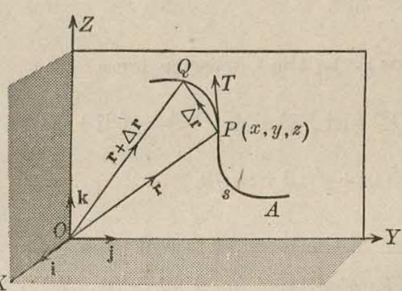


Fig. 6.2. Tangent to a curve at a point

We take the points P and Q corresponding to the parametric values t and $t+\triangle t$. Then we have

$$\mathbf{r}=\overrightarrow{OP}=\mathbf{f}(t)\;;\quad \mathbf{r}+\triangle\mathbf{r}=\overrightarrow{OQ}=\mathbf{f}(t+\triangle t)$$

so that $\overrightarrow{PQ}=\triangle\mathbf{r}=\mathbf{f}(t+\triangle t)-\mathbf{f}(t)$.

Dividing by $\triangle t$, the increment in t, we obtain

$$\frac{\overrightarrow{PQ}}{\triangle t}=\frac{\triangle \mathbf{r}}{\triangle t}=\frac{\mathbf{f}(t+\triangle t)-\mathbf{f}(t)}{\triangle t},$$

the limiting position of PQ is given by the vector

$$\lim_{\triangle t\to 0}\frac{\overrightarrow{PQ}}{\triangle t}=\lim_{\triangle t\to 0}\frac{\triangle \mathbf{r}}{\triangle t}=\lim_{\triangle t\to 0}\frac{\mathbf{f}(t+\triangle t)-\mathbf{f}(t)}{\triangle t}$$

That is, $\dfrac{d\mathbf{r}}{dt}$ or $\mathbf{f}'(t)$ is parallel to the tangent PT of the curve at P.

Next it is easy to think through that the equation

$$\mathbf{R}=\mathbf{r}+\lambda\frac{d\mathbf{r}}{dt} \text{ or } (\mathbf{R}-\mathbf{r})\times\frac{d\mathbf{r}}{dt}=\mathbf{0} \qquad \cdots \quad (1)$$

represents the equation of the tangent line PT, where λ is any arbitrary scalar and $\mathbf{r}$ is the position vector of P and $\mathbf{R}$ is the position vector of any point (current coördinate) on the tangent line.

Cartesian form : In the Cartesian form

$$\mathbf{r}=\overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\ ;\ \mathbf{f}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$$

so that $\dfrac{d\mathbf{r}}{dt}=x'(t)\mathbf{i}+y'(t)\mathbf{j}+z'(t)\mathbf{k}.$

The equation of the tangent line at P is then

$$\mathbf{R}=(X, Y, Z)=(x, y, z)+\lambda\,[x'(t), y'(t), z'(t)].$$

$$\text{or} \quad \frac{X-x}{x'(t)}=\frac{Y-y}{y'(t)}=\frac{Z-z}{z'(t)}=\lambda \qquad \cdots \qquad \cdots \quad (2)$$

where (X, Y, Z) is any point on the tangent line at $P\,(x, y, z)$.

Illustration. Find the equation of the tangent to the curve

$$x=t,\ y=t^2,\ z=\tfrac{2}{3}t^3,\ \text{ or, } \mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+\tfrac{2}{3}t^3\mathbf{k}$$

at the point $t=1$.

(i) *Vector form* : $\mathbf{R}=(\mathbf{i}+\mathbf{j}+\frac{2}{3}\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

For, at $t=1$, $\mathbf{r}$ is given by $\mathbf{i}+\mathbf{j}+\frac{2}{3}\mathbf{k}$.

Also $x'(t)=1$, $y'(t)=2t$, $z'(t)=2t^2$, so that

$$\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+2t^2\mathbf{k}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \text{ when } t=1.$$

and the current vector $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

(ii) *Cartesian form* : $\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-\frac{2}{3}}{2}$,

where (x, y, z) is any point on the tangent line at $P(1, 1, \frac{2}{3})$.

2. Tangent line at a point $P(\mathbf{r})$ of the curve $\mathbf{r}=\mathbf{f}(s)$: Unit tangent vector t.

We shall first prove the following important theorem :

Theorem. *If* $\mathbf{r}=\mathbf{f}(s)$ *be the vector equation of a smooth curve in terms of* $s=$ *arc* AP *as parameter, then*

$$\frac{d\mathbf{r}}{ds}=\mathbf{t} \qquad \cdots \qquad (3)$$

is a unit vector, tangent to the curve at P *and pointing in the direcion of increasing arcs.*

Proof. We again refer to Fig. 6.2.

Let $\quad \overrightarrow{OP}=\mathbf{r}=\mathbf{f}(s)\ ;\ \overrightarrow{OQ}=\mathbf{r}+\triangle\mathbf{r}=\mathbf{f}(s+\triangle s)$

Hence $\quad \triangle\mathbf{r}=\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\mathbf{f}(s+\triangle s)-\mathbf{f}(s)$ so that

$$\frac{\triangle\mathbf{r}}{\triangle s}=\frac{\overrightarrow{PQ}}{\text{arc } PQ}=\frac{\overrightarrow{PQ}}{\text{chord } PQ}\cdot\frac{\text{chord } PQ}{\text{arc } PQ}$$

Now it can be proved that the ratio of a chord of a smooth curve to the arc it subtends approaches unity as the arc approaches zero.

Therefore, when $Q\to P$ (*i.e.*, when $\triangle s\to 0$),

$$\frac{\text{chord } PQ}{\text{arc } PQ}\to 1$$

and $\overrightarrow{PQ}$/chord PQ, which is always a unit vector tends to the unit tangent vector $\mathbf{t}$; hence

$$\frac{d\mathbf{r}}{ds}=\mathbf{t}.$$

Note that $\mathbf{t}$ is a unit vector in the direction of the tangent to the curve at P in the sense of increasing s. In future we may refer to it simply as *unit tangent*.

If the position vector of P with reference to the rectangular axes through O is given by

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

then $$\mathbf{t}=\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}.$$

Hence the direction cosines of the tangent line at P are

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds} \qquad \cdots \qquad (4)$$

Next we write down the equation of the tangent line at P in

Vector form : $\mathbf{R}=\mathbf{r}+\lambda\dfrac{d\mathbf{r}}{ds}$ or $(\mathbf{R}-\mathbf{r})\times\dfrac{d\mathbf{r}}{ds}=\mathbf{0}$ $\cdots$ (5)

Cartesian form : $\dfrac{X-x}{dx/ds}=\dfrac{Y-y}{dy/ds}=\dfrac{Z-z}{dz/ds}$ $\cdots$ (6)

3. To obtain t when the curve is given in terms of a parameter other than s.

The following illustration will make the point clear :

Illustration. Given the space curve : $x=t,\ y=t^2,\ z=\frac{2}{3}t^3$.

The position vector $\mathbf{r}$ of any point is then given by

$$\mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+\tfrac{2}{3}t^3\mathbf{k}.$$

Then $$\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+2t^2\mathbf{k}.$$

But $$\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\,\frac{ds}{dt}=\mathbf{t}\,\frac{ds}{dt},$$

so that $$\left|\frac{ds}{dt}\right| = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\{1+(2t)^2+(2t^2)^2\}} = 1+2t^2.$$

$$\therefore \quad \mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\cdot\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}\Big/\frac{ds}{dt} = \frac{1}{1+2t^2}(\mathbf{i}+2t\mathbf{j}+2t^2\mathbf{k}).$$

6·31. Polar Coördinates.

In this article we propose to introduce a new parameter θ in representing the equation of a curve. Consider the relation

$$\mathbf{r} = \mathbf{R}(\theta) \qquad \cdots \qquad (1)$$

where $\mathbf{R}(\theta)$ is a variable unit vector in a *plane* making an angle of θ radians with some fixed line OX. If $\mathbf{R}(\theta)$ be drawn from the origin, its terminal point P will describe a circle of unit radius (See Fig. 6.3). Note that the arc $AP = s = \theta$ and A, the fixed point from which arc s is measured, is taken on OX. In this case the unit tangent vector

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta} = \frac{d\mathbf{R}}{d\theta} = \mathbf{R}(\theta + \tfrac{1}{2}\pi) = \mathbf{P}(\text{say}) \qquad \cdots \qquad (2)$$

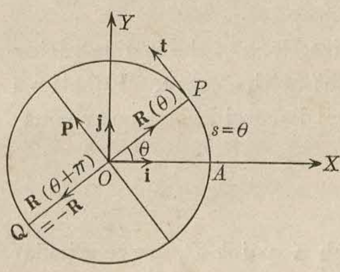


Fig. 6.3. Parameter θ in $\mathbf{r} = \mathbf{R}(\theta)$

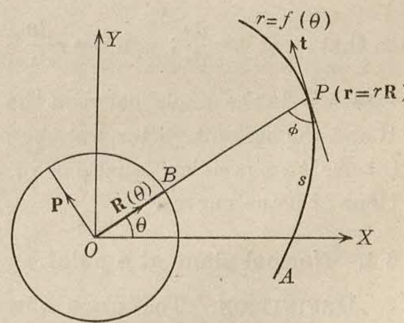


Fig. 6.4. Polar equation of a plane curve

This $\mathbf{P}$ is a unit vector perpendicular to $\mathbf{R}$ in the direction of increasing arcs (here *increasing angles*). Using the result we have also

$$\frac{d\mathbf{P}}{d\theta} = \mathbf{R}(\theta + \pi) = -\mathbf{R} = \overrightarrow{OQ} \qquad \cdots \qquad (3)$$

If the directions along OX and OY be given by the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we may write

$$\mathbf{R} = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$$

$$\therefore \quad \frac{d\mathbf{R}}{d\theta} = -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j} \qquad \ldots \qquad (4)$$

If $r = f(\theta)$ be the equation of a plane curve in polar coördinates, its vector equation is (See Fig. 6.4)

$$\mathbf{r} = r\mathbf{R}(\theta) \qquad \ldots \qquad (5)$$

Hence, on differentiation with respect to s, we have

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d}{ds}(r\mathbf{R}) = \frac{dr}{ds}\mathbf{R} + r\frac{d\mathbf{R}}{ds} = \frac{dr}{ds}\mathbf{R} + r\frac{d\mathbf{R}}{d\theta}\,\frac{d\theta}{ds}$$

and so $\quad \mathbf{t} = \dfrac{dr}{ds}\mathbf{R} + r\dfrac{d\theta}{ds}\,\mathbf{P} \qquad$ [using (2)]

Multiplying scalarly first with $\mathbf{R}$ and then with $\mathbf{P}$, we find

$$\mathbf{R}.\mathbf{t} = \frac{dr}{ds}\,; \quad \mathbf{P}.\mathbf{t} = r\frac{d\theta}{ds}.$$

so that $\quad \cos\phi = \dfrac{dr}{ds}\,; \quad \sin\phi = r\dfrac{d\theta}{ds} \qquad \ldots \qquad (6)$

where ϕ is the angle between the directions of the radius vector $\mathbf{R}$ and the tangent vector $\mathbf{t}$ at the point of the curve. Equations (6) are two familiar equations in the discussions on polar equations of plane curves.

6·4. Normal plane at a point.

DEFINITION. The plane through a point P, perpendicular to the tangent at P, is called the *normal plane* at that point.

If $\mathbf{R}$ denotes the position vector of any point on the normal plane at the point P whose position vector is $\mathbf{r}$ then the vector $\mathbf{R} - \mathbf{r}$ and the tangent vector $d\mathbf{r}/dt$ are at right angles.

Hence, $\quad (\mathbf{R} - \mathbf{r})\cdot\dfrac{d\mathbf{r}}{dt} = 0 \qquad \ldots \qquad (1)$

This gives the equation of the normal plane at the point $P(\mathbf{r})$.

In the cartesian notation the equation becomes

$$(X-x)\frac{dx}{dt}+(Y-y)\frac{dy}{dt}+(Z-z)\frac{dz}{dt}=0 \quad \cdots \quad (2)$$

Note that any line through P in this plane is *a normal* to the curve at P. As an illustration we consider :

Illustrative Example. Find the equation of the normal plane to the curve

$$x=t,\ y=t^2,\ z=\tfrac{2}{3}t^3$$

at the point $(1,\ 1,\ \tfrac{2}{3})$.

Here, $(\mathbf{R}-\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}=0$ reduces to

$$\{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(\mathbf{i}+\mathbf{j}+\tfrac{2}{3}\mathbf{k})\}\,.\,\{\mathbf{i}+2\mathbf{j}+2\mathbf{k}\}=0$$

i.e. $(x-1)+2(y-1)+2(z-\tfrac{2}{3})=0.$

6·5. Frenet's Formulæ : Principal Normal and Binormal.

In the following discussions we shall consider the equation of the curve in the form

$$\mathbf{r}=\mathbf{f}(s) \quad \cdots \quad (1)$$

where, as already explained, s denotes the length of the arc measured from some fixed point of the curve. We consider any point P of the curve which corresponds to the parameter s.

1. Unit tangent vector t.

In art. 6·3 we have shown that

$$\mathbf{t}=\frac{d\mathbf{r}}{ds} \quad \cdots \quad (2)$$

is the *unit vector*, in the direction of the tangent at a point P, where the parameter is s and it points in the direction of increasing s.

Further, we have established that the equation of the tangent line at P is given by

$$(\mathbf{R}-\mathbf{r})\times\mathbf{t}=\mathbf{0} \quad \cdots \quad (3)$$

2. Unit principal normal n.

Since the length of $\mathbf{t}$ is constant, its derivative $d\mathbf{t}/ds$, if not zero, must be perpendicular to $\mathbf{t}$ (See art 5·31) and therefore normal to the curve at P. A directed line through P in the direction of $d\mathbf{t}/ds$ is called the *principal normal* of the curve at P (Fig. 6.5). If $\mathbf{n}$ is a unit vector in the direction of the principal normal then we may write

$$\frac{d\mathbf{t}}{ds}=\kappa\mathbf{n} \qquad \cdots \quad (4)$$

where κ is a non-negative ($\geqslant 0$) scalar ; $\mathbf{n}$ is called the *unit principal normal* and κ is called the *curvature of the curve* at the point P.

We may now easily write down the equation of the principal normal at P in the form

$$(\mathbf{R}-\mathbf{r})\times\mathbf{n}=\mathbf{0}. \qquad \cdots \quad (5)$$

3. Unit binormal b.

We introduce another unit vector $\mathbf{b}$ defined by

$$\mathbf{b}=\mathbf{t}\times\mathbf{n} \qquad \cdots \quad (6)$$

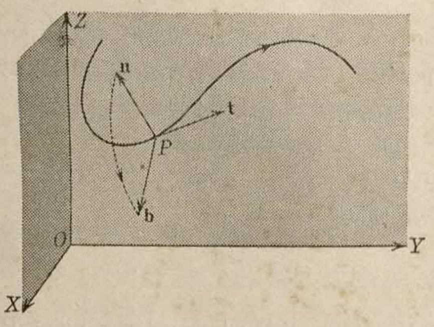


Fig. 6.5. The moving trihedral $\mathbf{t}, \mathbf{n}, \mathbf{b}$

so that the three vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ form a right-handed system of orthogonal unit vectors. A directed line through P in the direction of $\mathbf{b}$ is called *binormal* to the curve at P (Fig. 6.5). We may speak of $\mathbf{b}$ as the *unit binormal.*

The vector equation of the binormal at P is then

$$(\mathbf{R}-\mathbf{r})\times\mathbf{b}=\mathbf{0} \qquad \cdots \quad (7)$$

4. Moving Trihedral t, n, b.

At any specified point of a curve we have so far defined, *tangent*, *principal normal* and *binormal*, their respective directions

are given by **t, n** and **b**. These directions are mutually perpendicular. We may consider that **t, n, b** form a localised right-handed rectangular coördinate system at any particular point P of the curve. As P traverses this coördinate system moves and we speak of the moving trihedral **t n b** (Fig. 6.5).

5. Curvature : Radius of curvature and Circle of curvature.

We now show from simple geometric considerations the following theorem.

Theorem. *The curvature* κ *as defined by* (4) *is the arc-rate of turning of the tangent at* P.

Proof. Let the unit tangents at P and at a neighbouring point Q of the curve (1) be $\mathbf{t}$ and $\mathbf{t}+\Delta\mathbf{t}$ respectively, the parameters corresponding to them being s and $s+\Delta s$. We represent $\mathbf{t}$ and $\mathbf{t}+\Delta\mathbf{t}$ by $\overrightarrow{OT}$ and $\overrightarrow{OT'}$ so that $\overrightarrow{TT'}=\Delta\mathbf{t}$ (Fig. 6.6). Suppose $\Delta\theta$ be the angle TOT' ; $\Delta\theta$ thus measures the inclination of the tangents at P and Q and is equal to the *arc* TT' of a unit circle with centre at O. Then

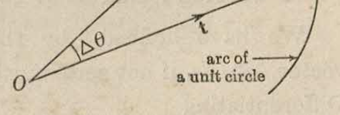


Fig. 6.6 To illustrate $\kappa = d\theta/ds$

$$\kappa = \left|\frac{d\mathbf{t}}{ds}\right| = \lim_{\Delta s\to 0}\left|\frac{\Delta\mathbf{t}}{\Delta s}\right| = \lim_{\Delta s\to 0}\left|\frac{\Delta\mathbf{t}}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\right| = \frac{d\theta}{ds},$$

since $\dfrac{\Delta t}{\Delta\theta}=\dfrac{\overrightarrow{TT'}}{\Delta\theta}=\dfrac{\overrightarrow{TT'}}{\text{arc } TT' \text{ of the unit circle with centre at } O}$.

and so tends to unity as $Q \to P$.

But $d\theta/ds$ is nothing but the rate at which the tangent turns with respect to the arc. Hence the proposition.

Observations. We have already stated that κ is a non-negative scalar. If $\kappa=0$ at all points of a curve, then by (4) $d\mathbf{t}/ds=\mathbf{0}$ and hence $\mathbf{t}$ is a unit vector of *constant direction* at every point of the curve *i.e.*, the curve is a straight line.

Conversly, for a straight line, $\mathbf{t}$ is a constant unit vector so that $d\mathbf{t}/ds=\mathbf{0}$ and hence $\kappa=0$. We *thus see that the only curves of zero curvature are straight lines.*

If $\kappa \neq 0$ we call the reciprocal of κ the *Radius of curvature* at the point and denote this reciprocal by ρ ; thus

$$\rho=\frac{1}{\kappa}. \qquad \cdots \quad (8)$$

The point C on the principal normal PN where $\overrightarrow{PC}=\rho\mathbf{n}$ is called the *Centre of curvature* at P.

The plane containing the *tangent* and *principal normal* (*i.e.*, determined by two unit vectors $\mathbf{t}$ and $\mathbf{n}$) is called the *Osculating plane* or *Plane of curvature* at the point P. In this plane if we draw a circle with centre C and radius ρ then the circle is called the *Circle of curvature* at P.

6. Torsion : Radius of torsion.

We have defined $\mathbf{b}$ by the relation (6). Since $\mathbf{b}$ is a unit vector, $d\mathbf{b}/ds$, if not zero, must be perpendicular to $\mathbf{b}$.
Differentiating

$$\mathbf{b}=\mathbf{t}\times\mathbf{n},$$

we have $\dfrac{d\mathbf{b}}{ds}=\dfrac{d\mathbf{t}}{ds}\times\mathbf{n}+\mathbf{t}\times\dfrac{d\mathbf{n}}{ds}=\mathbf{t}\times\dfrac{d\mathbf{n}}{ds}$

since $d\mathbf{t}/ds=\kappa\mathbf{n}$ and $\kappa\mathbf{n}\times\mathbf{n}=\mathbf{0}$.

Thus $d\mathbf{b}/ds$ is perpendicular to $\mathbf{t}$. But as stated before it is also perpendicular to $\mathbf{b}$, and must, therefore, be parallel to $\mathbf{n}$. We may then write,

$$\frac{d\mathbf{b}}{ds}=-\tau\mathbf{n} \qquad \cdots \quad (9)$$

where τ is a scalar, called the *torsion* of the curve at P. The minus sign in (9) has this purpose :

When $\tau > 0$, $\frac{d\mathbf{b}}{ds}$ has the direction of $-\mathbf{n}$; then, as P moves along the curve in a positive direction, $\mathbf{b}$ revolves about $\mathbf{t}$ in the same sense as a right-handed screw advancing in the direction of $\mathbf{t}$ (Fig. 6.5).

Further, we introduce the quantity σ defined by

$$\sigma = \frac{1}{\tau} \ (\tau \neq 0) \qquad \cdots \quad (10)$$

This quantity σ is called the *Radius of torsion.*

If $\tau = 0$ (identically), $d\mathbf{b}/ds = \mathbf{0}$ and $\mathbf{b}$ is a constant vector ; hence from

$$\mathbf{b}.\mathbf{t} = \mathbf{b}.\frac{d\mathbf{r}}{ds} = 0$$

we may write $$\mathbf{b}.(\mathbf{R} - \mathbf{r}) = 0$$

where $\mathbf{R}$ is the position vector of any point on the tangent line at the point $P(\mathbf{r})$ of the curve ; this proves that the curve lies in a plane normal to $\mathbf{b}$.

Conversely, for a plane curve $\mathbf{t}$ and $\mathbf{n}$ always lie on a fixed plane while $\mathbf{b}$ is a unit normal to that plane ; hence $d\mathbf{b}/ds = 0$ at all points where $\mathbf{n}$ is defined $(\kappa \neq 0)$ and $\tau = 0$. Thus the *only curves of zero torsion are planes.*

7. Frenet's Formulæ.

A set of relations involving the derivatives of the fundamental vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is known as *Frenet's formulae* (or *Serret-Frenet formulae*). We have already obtained the first two and now we shall obtain the third one of the following relations :

$$\left.\begin{aligned} \frac{d\mathbf{t}}{ds} &= \kappa\mathbf{n} \\ \frac{d\mathbf{b}}{ds} &= -\tau\mathbf{n} \\ \frac{d\mathbf{n}}{ds} &= -\kappa\mathbf{t} + \tau\mathbf{b} \end{aligned}\right\} \qquad \cdots \quad (11)$$

The three formulae (11) is known collectively as *Frenet's formulae.* The last result easily follows if we differentiate $\mathbf{n}=\mathbf{b}\times\mathbf{t}$ with respect to s ; thus

$$\frac{d\mathbf{n}}{ds}=\frac{d\mathbf{b}}{ds}\times\mathbf{t}+\mathbf{b}\times\frac{d\mathbf{t}}{ds}=-\tau\mathbf{n}\times\mathbf{t}+\mathbf{b}\times\kappa\mathbf{n}$$

$$=\tau\mathbf{b}+\kappa(-\mathbf{t})=-\kappa\mathbf{t}+\tau\mathbf{b}.$$

8. Three Fundamental Planes associated with every ordinary point of a curve.

With every ordinary point of a smooth curve we associate three mutually perpendicular lines, *viz.*

(i) Tangent line ; (ii) Principal normal and (iii) Binormal. They constitute the principal trihedron at a specified point of the curve. These three lines, taken in pairs, determine three mutually perpendicular planes (Fig. 6.7), namely

(iv) Osculating plane ; (v) Normal plane and (vi) Rectifying plane.

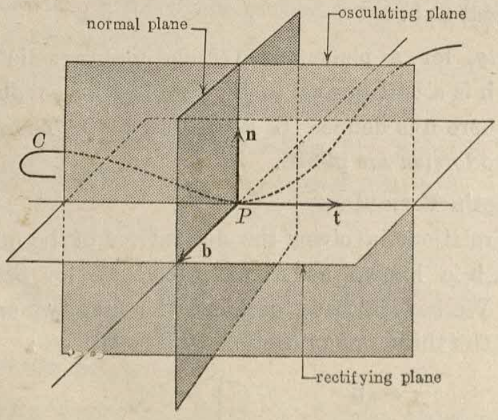


Fig. 6.7 Three Fundamental planes

The *Osculating plane* to a curve at a point P is the plane containing the tangent and principal normal at P.

The *Normal plane* is the plane through P perpendicular to the tangent; hence this plane contains the principal normal and binormal at the point P. *The Rectifying plane* is the plane through P which is perpendicular to the principal normal and hence it contains the tangent and binormal at P. (See Fig. 6.7).

The reader may now easily obtain the equations of these planes in the following forms where $\mathbf{R}$ is the position vector of any point on the plane and $\mathbf{r}$ is the position vector of a specified point of the curve :

Osculating plane : $(\mathbf{R}-\mathbf{r}).\mathbf{b}=0.$ ... (12)

Normal plane : $(\mathbf{R}-\mathbf{r}).\mathbf{t}=0.$ ... (13)

Rectifying plane : $(\mathbf{R}-\mathbf{r}).\mathbf{n}=0.$ ... (14)

6·51. Use of parameters other than the arc-length.

If a curve be represented in terms of a parameter t, other than the arc-length s, we shall use small dots over the symbols to represent derivations with respect to t. With the help of Frenet's formulæ we may compute the curvature κ and torsion τ of a curve whose parametric equation is

$$\left.\begin{array}{ll}\textbf{Vector form :} & \mathbf{r}=\mathbf{f}(t)\\ \textbf{Cartesian form :} & x=x(t),\quad y=y(t),\quad z=z(t)\end{array}\right\} \quad \cdots \quad (1)$$

On differentiating (1) three times, we get successively

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\,\frac{ds}{dt}=\mathbf{t}\,\dot{s} \quad \cdots \quad (2)$$

It follows, that $|\,\dot{\mathbf{r}}\,|=|\,\dot{s}\,|$ ... (3)

Again, $\therefore\ \ddot{\mathbf{r}}=\dfrac{d\mathbf{t}}{dt}\,\dot{s}+\mathbf{t}\,\ddot{s}=\dfrac{d\mathbf{t}}{ds}\,\dfrac{ds}{dt}\,\dot{s}+\mathbf{t}\,\ddot{s}$

$$=\kappa\mathbf{n}\,\dot{s}^2+\mathbf{t}\,\ddot{s} \quad \cdots \quad (4)$$

and lastly, $\dddot{\mathbf{r}} = \dot{\kappa}\mathbf{n}\dot{s}^2 + \kappa \dfrac{d\mathbf{n}}{dt}\dot{s}^2 + \kappa\mathbf{n}\, 2\dot{s}\,\ddot{s} + \dfrac{d\mathbf{t}}{dt}\ddot{s} + \mathbf{t}\dddot{s}$

$$= \dot{\kappa}\mathbf{n}\dot{s}^2 + \kappa \frac{d\mathbf{n}}{ds}\frac{ds}{dt}\dot{s}^2 + \kappa\mathbf{n}\, 2\dot{s}\,\ddot{s} + \frac{d\mathbf{t}}{ds}\frac{ds}{dt}\ddot{s} + \mathbf{t}\dddot{s}$$

$$= \dot{\kappa}\mathbf{n}\dot{s}^2 + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b})\dot{s}^3 + \kappa\mathbf{n}2\dot{s}\,\ddot{s} + \kappa\mathbf{n}\dot{s}\,\ddot{s} + \mathbf{t}\dddot{s}$$

$$= \dddot{s}\mathbf{t} + \dot{s}\,\ddot{s}\kappa\mathbf{n} + (2\dot{s}\,\ddot{s}\kappa + \dot{s}^2\dot{\kappa})\mathbf{n} + \dot{s}^3\kappa(-\kappa\mathbf{t} + \tau\mathbf{b})$$

$$= (\dddot{s} - \dot{s}^3\kappa^2)\mathbf{t} + (3\dot{s}\,\ddot{s}\kappa + \dot{s}^2\dot{\kappa})\mathbf{n} + \dot{s}^3\kappa\tau\,\mathbf{b} \quad \cdots \quad (5)$$

Hence, we obtain, $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \dot{s}^3\kappa\mathbf{b}$ $\cdots$ (6)

and $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}.\dddot{\mathbf{r}} = [\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \dot{s}^6\kappa^2\tau$ $\cdots$ (7)

It then follows from (6) that

$$\kappa = \frac{|\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,|}{\dot{s}^3} = \frac{|\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,|}{|\,\dot{\mathbf{r}}\,|^3} \quad [\text{ using (3) }] \quad \cdots \quad (8)$$

and from (7) it follows

$$\tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{\dot{s}^6\kappa^2} = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,|^2} \quad [\text{ using (6) }] \quad (9)$$

Thus when the equation of the curve is given in the form (1) then (8) and (9) would give the curvature and torsion at any point t of the curve.

If the positive direction on the curve is the direction in which t increases then $\dot{s} = ds/dt$ is positive and then (2) and (6) would lead us to conclude :

t, b, n *have the directions of* $\dot{\mathbf{r}}$, $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$, $(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \times \dot{\mathbf{r}}$ $\cdots$ (10)

The planes through a point $\mathbf{r}(t)$ of the curve and perpendicular to **t, b** and **n** are, as we know, called respectively, the normal, osculating, and rectifying planes to the curve. Using (10) we can easily obtain their equations.

Osculating Plane : $(\mathbf{R} - \mathbf{r}) \cdot \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 0.$

Normal Plane : $(\mathbf{R} - \mathbf{r}) \cdot \dot{\mathbf{r}} = 0.$

Rectifying Plane : $(\mathbf{R} - \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \times \dot{\mathbf{r}} = 0.$

We remember that $\mathbf{R}$ denotes the position vector of any point on the plane and $\mathbf{r}$ is the position vector of the point on the space curve with which the plane is associated.

6·6. Applications.

[Dashes denote derivations with respect to the parameter s ; dots denote differentiations with respect to a parameter other than s].

1. Prove the relations

$$\kappa = |\,\mathbf{r}''\,|, \ \mathbf{r}' \times \mathbf{r}'' = \kappa \mathbf{b},$$
$$\mathbf{r}''' = \kappa(\tau \mathbf{b} - \kappa \mathbf{t}) + \kappa' \mathbf{n}, \ [\,\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}'''\,] = \kappa^2 \tau.$$

Solution. Since $\mathbf{r}' = \mathbf{t}$, we have $\mathbf{r}'' = \mathbf{t}' = \kappa \mathbf{n}$ (Frenet's first formula) and hence $|\,\mathbf{r}''\,| = \kappa$ (*First result*).

Again $\mathbf{r}' \times \mathbf{r}'' = \mathbf{t} \times \kappa \mathbf{n} = \kappa \mathbf{b}$ whence follows the *Second result.*

Thirdly $\mathbf{r}''' = \kappa' \mathbf{n} + \kappa \mathbf{n}' = \kappa' \mathbf{n} + \kappa(-\kappa \mathbf{t} + \tau \mathbf{b})$ (Frenet's formula), which gives the *Third result.*

Lastly, consider $\mathbf{r}'' \times \mathbf{r}''' = \kappa \mathbf{n} \times \{\kappa' \mathbf{n} + \kappa(-\kappa \mathbf{t} + \tau \mathbf{b})\}$

$$= -\kappa^3 \mathbf{n} \times \mathbf{t} + \kappa^2 \tau\, \mathbf{n} \times \mathbf{b} = \kappa^2 \tau\, \mathbf{t} + \kappa^3 \mathbf{b}.$$

whence $\mathbf{r}' \cdot \mathbf{r}'' \times \mathbf{r}''' = \mathbf{t} \cdot (\kappa^2 \tau\, \mathbf{t} + \kappa^3 \mathbf{b}) = \kappa^2 \tau$ (*Fourth result*).

2. Show that for any curve $\mathbf{r} = \mathbf{f}(s)$,

$$[\,\mathbf{t}'\ \mathbf{t}''\ \mathbf{t}'''\,] = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right); \ [\,\mathbf{b}'\ \mathbf{b}''\ \mathbf{b}'''\,] = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

Left as an exercise for the students.

3. Darboux vector.

[We define the vector $\mathbf{d}$ by the relation,

$$\mathbf{d} = \tau \mathbf{t} + \kappa \mathbf{b}.$$

It represents the arc-rate of rotation of the moving trihedral $\mathbf{t}, \mathbf{n}, \mathbf{b}$ as the point P moves along the curve. The part $\kappa \mathbf{b}$ is the rate of turning about the binormal and is due to curvature. The part $\tau \mathbf{t}$ is the rate of turning about the tangent and is due to torsion. We call $\mathbf{d}$ as the **Darboux Vector** of the curve].

Prove that Frenet's formulæ take the form :

$$\mathbf{t}' = \mathbf{d} \times \mathbf{t}, \quad \mathbf{n}' = \mathbf{d} \times \mathbf{n}, \quad \mathbf{b}' = \mathbf{d} \times \mathbf{b}$$

where $\mathbf{d}$ is the *Darboux vector* of the curve, dashes denoting derivations with respect to s. Also show that

$$\mathbf{d} = \mathbf{n} \times \mathbf{n}' = \mathbf{n} \times (\mathbf{d} \times \mathbf{n}).$$

Solution. Since $\mathbf{d} = \tau \mathbf{t} + \kappa \mathbf{b}$, we have

$$\mathbf{d} \times \mathbf{t} = \tau \mathbf{t} \times \mathbf{t} + \kappa \mathbf{b} \times \mathbf{t} = \kappa \mathbf{b} \times \mathbf{t} = \kappa \mathbf{n} = \frac{d\mathbf{t}}{ds} = \mathbf{t}'.$$

Similarly other relations for $\mathbf{n}'$ and $\mathbf{b}'$ can be obtained.

Again, since $\mathbf{n} \times (\mathbf{d} \times \mathbf{n}) = (\mathbf{n} \,.\, \mathbf{n})\, \mathbf{d} - (\mathbf{n} \,.\, \mathbf{d})\, \mathbf{n}$

and $\mathbf{n} \,.\, \mathbf{d} = \mathbf{n} \,.\, (\tau \mathbf{t} + \kappa \mathbf{b}) = \tau\, \mathbf{n} \,.\, \mathbf{t} + \kappa\, \mathbf{n} \,.\, \mathbf{b} = 0$

the last result follows easily.

4. For a plane curve

$$x = x(t), \quad y = y(t)$$

obtain the expressions for κ and τ.

Solution. We have

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}, \quad \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}.$$

Now from (8) and (9) of art. 6·51 we may easily obtain

$$\kappa = \frac{|\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,|}{|\,\dot{\mathbf{r}}\,|^3} = \frac{|\,\dot{x}\ddot{y} - \dot{y}\ddot{x}\,|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \quad \tau = 0.$$

In particular, if the curve has the Cartesian equation $y = f(x)$, we can regard x as the parameter : $x = t$, $y = f(t)$. Then from above, κ is given by

$$\kappa = \frac{|\,\ddot{y}\,|}{(1 + \dot{y}^2)^{3/2}},$$

where dots represent derivations with respect to x.

5. If the curve has the polar equation $r = f(\theta)$, we regard the parameter $t = \theta$ (See art. 6·31). Then

$$\mathbf{r} = r\, \mathbf{R}(\theta) = r\, \mathbf{R} \text{ (say)} ; \quad \dot{\mathbf{r}} = \dot{r}\, \mathbf{R} + r\, \mathbf{P}, \quad \ddot{\mathbf{r}} = (\ddot{r} - r)\, \mathbf{R} + 2\dot{r}\, \mathbf{P}.$$

(For explanations of the symbols $\mathbf{R}$ and $\mathbf{P}$ see art 6·31).

and since, $|\dot{\mathbf{r}}| = (r^2 + \dot{r}^2)^{1/2}$, $|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = |2\dot{r}^2 - r(\ddot{r} - r)|$

we have from (8) of art. 6·51,

$$\kappa = \frac{|r^2 + 2\dot{r}^2 - r\ddot{r}|}{(r^2 + \dot{r}^2)^{3/2}}.$$

6. Twisted Cubic. For the *twisted cubic*

$$x = 2t, \ y = t^2, \ z = \tfrac{1}{3}t^3$$

find $\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau$ and the equations of the osculating plane, normal plane and rectifying plane at the point $t = 1$. Find also the length of the curve measured from $t = 0$.

Solution. We have

$$\mathbf{r}(t) = (2t, t^2, \tfrac{1}{3}t^3).$$

Hence, $\dot{\mathbf{r}} = (2, 2t, t^2)$, $\ddot{\mathbf{r}} = (0, 2, 2t)$; $\dddot{\mathbf{r}} = (0, 0, 2)$.

At the point $t = 1$, we then have

$$\mathbf{r} = (2, 1, \tfrac{1}{3}) \ ; \ \dot{\mathbf{r}} = (2, 2, 1) \ ; \ \ddot{\mathbf{r}} = (0, 2, 2) \ ; \ \dddot{\mathbf{r}} = (0, 0, 2)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (2, -4, 4), \ \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \cdot \dddot{\mathbf{r}} = 8.$$

Hence from (10) of art. 6·51, we obtain

$$\mathbf{t} = \tfrac{1}{3}(2, 2, 1), \ \mathbf{b} = \tfrac{1}{3}(1, -2, 2), \ \mathbf{n} = \mathbf{b} \times \mathbf{t} = \tfrac{1}{3}(-2, 1, 2)$$

Then from (8) and (9) of art. 6·51, we deduce

$$\kappa = 2/9, \ \ \tau = 2/9.$$

The equation of the *osculating plane* in vector form is

$$(\mathbf{R} - \mathbf{r}) \cdot \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 0$$

which reduces to the cartesian form

$$2(x - 2) + (y - 1)(-4) + (z - \tfrac{1}{3})4 = 0$$

or, $x - 2y + 2z = 2/3.$

Similarly the equation of the *normal plane* is

$$2(x - 2) + 2(y - 1) + (z - \tfrac{1}{3}) = 0 \ ;$$

and the equation of the *rectifying plane* is

$$-12(x-2)+6(y-1)+12(z-\tfrac{1}{3})=0$$

$$[\text{ since } (\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\times\dot{\mathbf{r}}=(-12,\ 6,\ 12)\ ].$$

Lastly since $\dot{\mathbf{r}}=\dot{s}\mathbf{t}$ or $\dot{s}=|\dot{\mathbf{r}}|$, the length of the curve measured from $t=0$ upto any point t is

$$L=\int_0^t |\dot{\mathbf{r}}|\,dt=\int_0^t \sqrt{4+4t^2+t^4}\,dt.$$

Examples. VI

[Dashes denote derivations with respect to the arc-length parameter s; dots denote derivations with respect to other parameters like t, θ etc.]

1. (*a*) Given a curve in the parametric form as

$$x=t, \qquad y=t^2, \qquad z=\tfrac{2}{3}t^3$$

so that in the form of a vector function we may write its equation as

$$\mathbf{r}=(t,\ t^2,\ \tfrac{2}{3}t^3).$$

Now obtain at any point t the following :

$$\dot{\mathbf{r}},\ \dot{s},\ \mathbf{t},\ \dot{\mathbf{t}},\ \mathbf{t}',\ \kappa,\ \rho,\ \mathbf{n},\ \mathbf{b},\ \dot{\mathbf{b}},\ \mathbf{b}',\ \tau,\ \sigma.$$

At the point $t=1$, obtain the equations of tangent, principal normal, binormal, osculating plane, normal plane and rectifying plane.

(*b*) A space curve is given in terms of arc-length parameter s by the equations,

$$x=\tan^{-1}s,\ y=(1/\sqrt{2})\log(s^2+1),\ z=s-\tan^{-1}s$$

Find $\quad \mathbf{t},\ \mathbf{n},\ \mathbf{b},\ \kappa,\ \tau,\ \rho,\ \sigma.$

2. Find the osculating plane, curvature and torsion at any point of the curve :

$$x=a\cos 2t,\ y=a\sin 2t,\ z=2a\sin t.$$

3. The *circular helix* is a curve drawn on the surface of a circular cylinder and cutting the generators at a constant angle (art. 6·2, Ex. iii). If the origin be taken on the axis of the cylinder, the equation of the helix on the circular cylinder of radius a is given by

$$x = a \cos \theta,\ y = a \sin \theta,\ z = b\theta, \quad (b = \text{constant}),$$

or, $$\mathbf{r} = (a \cos \theta,\ a \sin \theta,\ b\theta).$$

Verify that for this circular helix,

$$\kappa = a/(a^2 + b^2),\ \tau = b/(a^2 + b^2)$$

and that the length of the curve measured from $t = 0$ to some point t is $t\sqrt{a^2 + b^2}$.

In particular, take $b = a \cot \beta$ (β = constant) and then obtain the expressions for $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and see that

$$a\kappa = \sin^2 \beta,\ a\tau = \sin \beta \cos \beta.$$

4. Obtain κ and τ for the following curves :

(i) $\mathbf{r} = (\log \cos \theta,\ \log \sin \theta,\ \theta\sqrt{2})$;

(ii) $\mathbf{r} = a(\cos \theta,\ \sin \theta,\ \log \sec \theta)$;

(iii) $\mathbf{r} = (e^t,\ e^{-t},\ \sqrt{2}t)$;

(iv) $\mathbf{r} = a(3t - t^3,\ 3t^2,\ 3t + t^3)$.

5. At the points $t = 0$ and $t = 1$ of the twisted cubic

$$\mathbf{r} = (3t,\ 3t^2,\ 2t^3)$$

find $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, κ, τ. Find also the equations of the normal and osculating planes.

6. Find the equation of the osculating plane at any point t of the curve

$$x = 2 \log t,\ y = 4t,\ z = 2t^2 + 1.$$

7. If the tangent to a curve makes a constant angle α with a fixed line, then $\kappa = \pm \tau \tan \alpha$ and conversely, show that if κ/τ is constant, the tangent makes a constant angle with a fixed direction.

8. Helics : A *helix* is a twisted curve whose tangent makes a constant angle with a fixed line called its *axis*. If the unit vector $\mathbf{e}$ has the direction of the axis, the defining equation of the helix is $\mathbf{e}.\mathbf{t} = \cos\alpha\,(0 < \alpha < \frac{1}{2}\pi)$. Prove the following statements for a helix :

(i) the principal normal is always perpendicular to the axis ;

(ii) the Darboux vector $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$ has a constant direction (always parallel to the axis) ;

(iii) ratio of curvature to torsion is constant : $\kappa/\tau = \pm \tan\alpha$.

9. Prove that the Darboux vector $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$ is *constant* if κ and τ are constants and that $\mathbf{d}$ has a *fixed direction* if κ/τ is constant.

10. Prove that a necessary and sufficient condition that a curve, $r = f(s)$ be a helix is that $\mathbf{r}''.\mathbf{r}''' \times \mathbf{r}'''' = 0$.

11. If κ is the curvature of a curve, then that of its projection on a plane inclined at an angle β to the plane of curvature is $\kappa\cos\beta$ if the plane is parallel to the tangent and $\kappa\sec^2\beta$ if it is parallel to the principal normal.

12. Prove that circular helics are the only twisted curves for which κ and τ are both constants.

13. Show that the locus of the feet of the perpendiculars from the origin to the tangents to the curve

$$\mathbf{r} = (a\cos\theta,\ a\sin\theta,\ a\theta)$$

is a curve which lies completely on the hyperboloid

$$x^2 + y^2 - z^2 = a^2.$$

14. If the tangent and binormal at a point of a curve make angles θ and ϕ respectively with a fixed direction, show that

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}.$$

15. Two curves are called *parallel* if a plane normal to one at any point P is also normal to the other at the corresponding point P_1. Prove that the distance PP_1 is constant.

Prove that $\kappa/\tau = \kappa_1/\tau_1$ at corresponding points of the curves.

16. On a plane sheet of paper a circle of radius r_1 is drawn and then it is folded in the form of a cylinder of radius r_2.

Show that for the new curve,

$$\kappa^2 = \frac{1}{r_1^2} + \frac{1}{r_2^2} \cos^4 \frac{s}{r_1},$$

where s is the length of the arc measured from a certain point.

17. The osculating plane at every point of a curve touches a fixed sphere ; show that the plane through the tangent perpendicular to the principal normal passes through the centre of the sphere.

18. Find the torsion of the curve

$$x = (2t+1)/(t-1), \ y = t^2/(t-1), \ z = t+2.$$

Explain your answer.

19. The curve $\mathbf{r} = \mathbf{f}(s)$ has the parametric equations

$$x = x(s), \quad y = y(s), \quad z = z(s).$$

For this curve deduce that

$$\text{(i)} \ \ \rho = \{(x'')^2 + (y'')^2 + (z'')^2\}^{-\frac{1}{2}}$$

$$\text{(ii)} \ \ \tau/\rho^2 = \mathbf{r}'.\mathbf{r}'' \times \mathbf{r}'''.$$

20. Plane curves :

(*a*) *Circle.* A circle of radius a about the origin has the position vector.

$\therefore$ $\mathbf{r} = a\mathbf{R}(\theta)$ where $\theta = s/a$ radians (Fig. 6.3)

Differentiating with respect to s, obtain

$$\mathbf{t} = \mathbf{P}, \ \kappa\mathbf{n} = -\mathbf{R}/a$$

and then see that

$$\mathbf{n} = -\mathbf{R}, \ \kappa = 1/a, \ \rho = a.$$

Show that the only plane curves of constant non-zero curvature are circles.

(*b*) *Ellipse*. Find κ and ρ of the ellipse,

$$x = a \cos t, \ y = b \sin t$$

Prove that the normal to the ellipse at any point P bisects the angle between the focal radii.

(*c*) *Intrinsic equation*. Let ψ be the angle from a fixed line in the plane to the tangent at a point P of a plane curve, taken in the positive sense and let s be the arc-length of the curve measured from a fixed point. Deduce

$$\kappa = d\psi/ds, \ \rho = ds/d\psi.$$

[*Intrinsic equation of a plane curve is defined as the relation between* s *and* ψ.]

Find ρ for a circle whose *intrinsic equation* is $r = a\psi$, where ψ is measured from the tangent at the point $s = 0$.

21. Evolute and Involute.

At any point P of a *plane curve* where $\kappa \neq 0$, *the centre of curvature* P_1 is given by

$$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} \qquad \cdots \quad (1)$$

The locus of P_1 is called the *evolute* of the curve. If $s = AP$ and $s_1 = A_1P_1$ denote the corresponding arc-lengths on the curve Γ and its evolute Γ_1, we find, taking s-derivative of (1)

$$\mathbf{t}_1 \frac{ds_1}{ds} = \mathbf{t} + \rho \frac{d\mathbf{n}}{ds} + \mathbf{n} \frac{d\rho}{ds} = \mathbf{n} \frac{d\rho}{ds},$$

since the second formula of Frenet becomes $\mathbf{n}' = -k\mathbf{t} = -\mathbf{t}/\rho$ as $\tau = 0$ for a plane curve.

Choose the positive direction on Γ_1 so that $\mathbf{t}_1 = \mathbf{n}$; then

$$\frac{ds_1}{ds} = \frac{d\rho}{ds},$$

whence $s_1 = \rho + \text{constant}.$

and since $\triangle s_1 = \triangle \rho$, an arc of the evolute is equal to the difference in the values of ρ at its end points.

These properties show that a curve Γ may be traced by the end P of a taut string unwound from its evolute Γ_1; the string is always tangent to Γ_1 and its free portion is equal to ρ. From this point of view, Γ is called the *involute* of Γ_1.

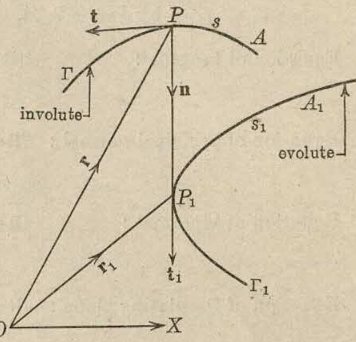


Fig. 6.8. Evolute and Involute of a plane curve

Since $\mathbf{t}_1 = \mathbf{n}_1$, $\psi_1 = (\widehat{\mathbf{i}, \mathbf{t}_1}) = (\widehat{\mathbf{i}, \mathbf{t}}) + \frac{1}{2}\pi = \psi + \frac{1}{2}\pi$;

$\therefore$ radius of curvature of the evolute is

$$\rho_1 = \frac{ds_1}{d\psi_1} = \frac{d\rho}{d\psi} = \frac{d^2 s}{d\psi^2}.$$

Hints and Answers

1. (*a*) $\dot{\mathbf{r}} = (1, 2t, 2t^2)$; $\dot{s} = |\dot{\mathbf{r}}| = \sqrt{(1+4t^2+4t^4)} = 1+2t^2$.

$$\mathbf{t} = \mathbf{r}' = \dot{\mathbf{r}}/\dot{s} = \frac{1}{1+2t^2}(1, 2t, 2t^2); \ \dot{\mathbf{t}} = \frac{1}{(1+2t^2)^2}(-4t, 2-4t^2, 4t).$$

$$\mathbf{t}' = \dot{\mathbf{t}}/\dot{s} = \frac{1}{(1+2t^2)^3}(-4t, 2-4t^2, 4t); \ \kappa = |\mathbf{t}'| = \frac{2}{(1+2t^2)^2}.$$

$$\rho = \tfrac{1}{2}(1+2t^2)^2; \ \mathbf{n} = \frac{1}{\kappa}\mathbf{t}' = \frac{1}{1+2t^2}(-2t, 1-2t^2, 2t);$$

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} = \frac{1}{1+2t^2}(2t^2, -2t, 1); \ \dot{\mathbf{b}} = \frac{1}{(1+2t^2)^2}(4t, 4t^2-2, -4t);$$

$$\mathbf{b}'=\dot{\mathbf{b}}/\dot{s}=\frac{1}{(1+2t^2)^3}(4t,\ 4t^2-2,\ -4t)\ ;\quad \tau=|\mathbf{b}'|=\frac{2}{(1+2t^2)^2}$$

$$\sigma=\tfrac{1}{2}(1+2t^2)^2.$$

At the point $t=1$, $\mathbf{r}=(1,\ 1,\ 2/3)$; $\mathbf{t}=\frac{1}{3}(1,\ 2,\ 2)$; $\mathbf{n}=\frac{1}{3}(-2,\ -1,\ 2)$; $\mathbf{b}=\frac{1}{3}(2,\ -2,\ 1)$.

Equation of tangent : $(\mathbf{R}-\mathbf{r})\times\mathbf{t}=\mathbf{0}$;

$$(x-1)=(y-1)/2=(z-2/3)/2.$$

Equation of principal normal : $(\mathbf{R}-\mathbf{r})\times\mathbf{n}=\mathbf{0}$;

$$(x-1)/(-2)=(y-1)/(-1)=(z-2/3)/2.$$

Equation of binormal : $(\mathbf{R}-\mathbf{r})\times\mathbf{b}=\mathbf{0}$;

$$(x-1)/2=(y-1)/(-2)=(z-2/3)/1.$$

Equation of Osculating plane : $(\mathbf{R}-\mathbf{r})\cdot\mathbf{b}=0$;

$$2(x-1)-2(y-1)+(z-2/3)=0.$$

Equation of normal plane : $(\mathbf{R}-\mathbf{r})\cdot\mathbf{t}=0$;

$$(x-1)+2(y-1)+2(z-2/3)=0.$$

Equation of Rectifying plane : $(\mathbf{R}-\mathbf{r})\cdot\mathbf{n}=0$;

$$-2(x-1)-(y-1)+2(z-2/3)=0.$$

(*b*) $\mathbf{t}=\dfrac{1}{s^2+1}(1,\ \sqrt{2}\,s,\ s^2)$; $\mathbf{n}=\dfrac{1}{s^2+1}(-\sqrt{2}\,s,\ 1-s^2,\ \sqrt{2}\,s)$;

$\mathbf{b}=\dfrac{1}{s^2+1}(s^2,\ -\sqrt{2}\,s,\ 1)$; $\kappa=\sqrt{2}/(s^2+1)=\tau$;

$\sigma=\rho=(s^2+1)/\sqrt{2}$.

2. Osculating plane : $(\sin t+\sin 2t\ \cos t)\,x-(\cos t+\cos t\ \cos 2t)\,y+2z=3a\sin t$.

Curvature : $\dfrac{1}{2a}\sqrt{(5+3\cos^2 t)/(1+\cos^2 t)^3}$,

Torsion : $3/a\,(5\sec t+3\cos t)$.

3. (*Particular case when* $b=a\cot\beta$). $\mathbf{t}=\sin\beta\,(-\sin\theta,\ \cos\theta,\ \cot\beta)$; $\mathbf{n}=(-\cos\theta,\ -\sin\theta,\ 0)$; $\mathbf{b}=\sin\beta\,(\sin\theta\cot\beta,\ -\cos\theta\cot\beta,\ 1)$.

4. (i) $\kappa=\sqrt{2}\sin\theta\cos\theta=-\tau$; (ii) $a\kappa=\cos\theta\sqrt{(1+\cos^2\theta)}$, $a\tau=\sin\theta\cos\theta\,(2+\cos^2\theta)/(1+\cos^2\theta)$. (iii) $\kappa=\sqrt{2}\,(e^t+e^{-t})^2=-\tau$.

(iv) $\kappa=1/3a\,(1+t^2)^2=\tau$.

5. $\mathbf{t}=(1, 0, 0)$; $\mathbf{n}=(0, 1, 0)$; $\mathbf{b}=(0, 0, 1)$ at the point $t=0$.

$\mathbf{t}=\frac{1}{3}(1, 2, 2)$; $\mathbf{n}=\frac{1}{3}(-2, -1, 2)$; $\mathbf{b}=\frac{1}{3}(2, -2, 1)$ at the point $t=1$.

$\kappa=\tau=2/3$; $\kappa=\tau=2/27$ in the two respective cases.

At $t=0$, the two planes are $x=0$, $z=0$.

At $t=1$, the planes are $x+2y+2z=13$, $2x-2y+z=2$.

6. $2(x-2\log t)-(2/t)(y-4t)+(1/t^2)(z-2t^2-1)=0$.

7. Let $\mathbf{e}$ be the unit vector parallel to the fixed line so that by the given condition $\mathbf{t}\,.\,\mathbf{e}=\cos\alpha$. Differentiating we get $\mathbf{t}'\,.\,\mathbf{e}=0$. By Frenet's formula this reduces to $\kappa\mathbf{n}\,.\,\mathbf{e}=0$ *i.e.*, $\mathbf{n}\,.\,\mathbf{e}=0$. Thus $\mathbf{n}$ is perpendicular to $\mathbf{e}$ and the vectors $\mathbf{b}, \mathbf{t}, \mathbf{e}$ are coplanar, so that $\mathbf{b}\,.\,\mathbf{e}=\pm\sin\alpha$. Differentiating $\mathbf{n}\,.\,\mathbf{e}=0$ and using Frenet's formula we obtain $\mathbf{n}'\,.\,\mathbf{e}=0$, or $-(\kappa\mathbf{t}+\tau\mathbf{b})\,.\,\mathbf{e}=0$, or $\kappa\cos\alpha\pm\tau\sin\alpha=0$ or $\kappa=\pm\tau\tan\alpha$.

Conversely, let $\kappa=a\tau$ where a is some scalar constant. Now $\mathbf{t}'=(\kappa/\tau)\,\mathbf{b}'=a\mathbf{b}'$. Integrating we get $\mathbf{t}=a\mathbf{b}+\mathbf{c}$ where $\mathbf{c}$ is the constant vector of integration. Take scalar product with $\mathbf{t}$ and obtain $\mathbf{t}\,.\,\mathbf{c}=1$. Hence the tangent makes a constant angle with the direction of the fixed vector $\mathbf{c}$.

14. $\mathbf{t}\,.\,\mathbf{c}=\cos\theta$, $\mathbf{b}\,.\,\mathbf{c}=\cos\phi$, $\mathbf{c}$ is the unit vector in the fixed direction. Differentiate the two relations.

15. Suppose the arc lengths s and s_1 on two curves increase in the same direction. Then $\mathbf{t}_1=\mathbf{t}$ at corresponding points and s_1' is positive, Differentiating $\mathbf{t}_1=\mathbf{t}$ *w.r.* to s we have $\kappa_1\mathbf{n}_1 s_1'=\kappa\,\mathbf{n}$. Consequently $\mathbf{n}_1=\mathbf{n}$ and $s_1'=\kappa/\kappa_1$. Also $\mathbf{b}_1=\mathbf{t}_1\times\mathbf{n}_1=\mathbf{t}\times\mathbf{n}=\mathbf{b}$ and differentiation then gives $\tau_1\mathbf{n}_1 s_1'=\tau\,\mathbf{n}$. Thus $\tau/\tau_1=s_1'=\kappa/\kappa_1$.

The distance PP_1 between corresponding points is constant. For the derivations *w.r. to s* of $(\mathbf{r}_1-\mathbf{r})^2$ is $2(\mathbf{r}_1-\mathbf{r})\,.\,(\mathbf{t}_1 s_1'-\mathbf{t})=0$, since $\mathbf{r}_1-\mathbf{r}$ is parallel to the normal plane and therefore perpendicular to $\mathbf{t}$ and $\mathbf{t}_1$.

Otherwise. See that $\mathbf{r}_1=\mathbf{r}+\alpha\mathbf{n}+\beta\mathbf{b}$ where $\mathbf{r}, \mathbf{n}, \mathbf{b}$ refer to the first curve and α and β are scalars; now show that $d(\alpha^2+\beta^2)/ds=0$.

17. Let $\mathbf{a}$ be the centre and h, the radius of the fixed sphere. The Osculating plane $(\mathbf{R}-\mathbf{r})\,.\,\mathbf{b}=0$ will touch the sphere if $(\mathbf{a}-\mathbf{r})\,.\,\mathbf{b}=h$. Deriva-

tions *w.r. to s* will give $(\mathbf{a}-\mathbf{r}) \cdot \mathbf{b}'-\mathbf{r}' \cdot \mathbf{b}=0$ ($\mathbf{a}$, h being constants) whence $(\mathbf{a}-\mathbf{r}) \cdot \tau\mathbf{n}-\mathbf{t} \cdot \mathbf{b}=0$, *i.e.*, $(\mathbf{a}-\mathbf{r}) \cdot \mathbf{n}=0$ which shows that the plane $(\mathbf{R}-\mathbf{r}) \cdot \mathbf{n}=0$ through the tangent and perpendicular to $\mathbf{n}$ also passes through $\mathbf{a}$, the centre of the fixed sphere.

18. $\tau=0$; this means that the curve lies on a plane (in fact, see that the plane is $x-3y+3z=5$).

20. (*a*) *Second part* . $d\mathbf{n}/ds=-\kappa\, d\mathbf{r}/ds$ or $d(\rho\mathbf{n}+\mathbf{r})\,/\,ds=0$ or $\rho\mathbf{n}+\mathbf{r}=\mathbf{c}$ *i.e.*, $|\,\mathbf{r}-\mathbf{c}\,|^2=\rho^2$, equation of a circle of radius ρ whose centre is $\mathbf{c}$.

(*b*) $\kappa=ab/(a^2 \sin^2 t+b^2 \cos^2 t)^{3/2}=1/\rho$.

If r_1 and r_2 are the focal distances of P, then $r_1+r_2=2a$, the major axis. Derivations *w.r. to s* give $r_1'+r_2'=0$ *i.e.*, $(\mathbf{R}_1+\mathbf{R}_2) \cdot \mathbf{t}=0$,

[See art. 6·31. equation (6)].

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are unit vectors along the focal radii.

Thus $\mathbf{R}_1+\mathbf{R}_2$ is normal to the ellipse at P and bisects the interior angle between the focal radii at P.

(*c*) $d\mathbf{t}/ds=(d\mathbf{t}/d\psi) \cdot (d\psi/ds)=(\mathbf{b}\times\mathbf{t})(d\psi/ds)=\mathbf{n}(d\psi/ds)$.
But Frenet's formula gives $d\mathbf{t}/ds=\kappa\mathbf{n}$; hence $\kappa=d\psi/ds$ and so $\rho=ds/d\psi$.

7

Applications of Vector Calculus in Mechanics

7·1. Introduction.

This chapter will be devoted to a concise treatment of the elementary principles of Mechanics through the use of Calculus of Vectors. The adoption of Vector Analysis in the study of Mechanics is urged on the grounds of naturalness, simplicity and directness.

Mechanics is generally studied under two broad headings—*Kinematics* and *Kinetics*. The first is concerned with the geometry of the motion without any reference to the forces which cause the motion and the second deals with the action of forces on the motion of the bodies.

We should carefully note in the following discussions that a body in motion is associated with a suitable frame of reference which, if not explicitly stated, is to be considered implied.

7·2. Kinematics of a particle.

Velocity.

DEFINITION. The *velocity* of a particle relative to a suitable frame of reference is the time-rate of change of the position vector $\mathbf{r}$, of the particle relative to the given frame of reference.

Thus with reference to a frame if a particle P has at any instant the position vector $\mathbf{r}=\overrightarrow{OP}$ and if during an interval $\triangle t$, the increment in $\mathbf{r}$ be $\triangle \mathbf{r}$ then $\triangle \mathbf{r}/\triangle t$ is the average velocity of P relative to O during the interval $\triangle t$. Hence

$$\lim_{\triangle t\to 0}\frac{\triangle \mathbf{r}}{\triangle t}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\mathbf{v} \qquad \cdots \quad (1)$$

is the velocity of P at that instant t (*instantaneous velocity of P*).

Clearly, the motion of P is given by $\mathbf{r}$ as a vector function of the scalar variable t (time), say,

$$\mathbf{r}=\mathbf{f}(t) \qquad \cdots \quad (2)$$

The locus of P, called its *trajectory* is the curve whose vector equation is given by (2), t being regarded as a parameter. Again, if P describes the curve (2) then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\;\frac{ds}{dt}=\mathbf{t}\frac{ds}{dt}=\mathbf{t}v \qquad \cdots \quad (3)$$

where $\mathbf{t}$ is the unit tangent vector to the curve and $v=|\mathbf{v}|=\dfrac{ds}{dt}$ is called the *speed* of the particle (s is, as usual, the arc-length measured from a fixed point on the curve). Hence

The velocity at P is a vector $\mathbf{v}$ *whose direction is same as that of the tangent to the path at P and sense is the same or opposite to that of the unit tangent vector* $\mathbf{t}$, *according as ds/dt is positive or negative i.e.. according as s increases with the time t. By the length or magnitude of the velocity vector* $\mathbf{v}$ *we mean the speed* $v=ds/dt$ *of the particle at P.*

Acceleration.

DEFINITION, It is the time-rate of change of velocity. Thus acceleration vector $\mathbf{a}$ of a particle P at some instant t is given by

$$\mathbf{a}=\lim_{\Delta t\to 0}\;\frac{\Delta\mathbf{v}}{\Delta t}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=\frac{d^2\mathbf{r}}{dt^2} \qquad \cdots \quad (4)$$

Differentiating (3) with respect to t we obtain

$$\frac{d\mathbf{v}}{dt}=\frac{d\mathbf{t}}{dt}\,v+\mathbf{t}\,\frac{dv}{dt} \qquad \cdots \quad (5)$$

Now the unit tangent vector $\mathbf{t}$ may be regarded as a function of the arc length s of the trajectory and then

$$\frac{d\mathbf{t}}{dt}=\frac{d\mathbf{t}}{ds}\;\frac{ds}{dt}=(\kappa\;\mathbf{n})v=\frac{v}{\rho}\,\mathbf{n} \qquad \text{[Frenet's First Formula]}$$

where $\mathbf{n}$ is the unit principal normal and κ is the curvature and ρ is the radius of curvature. Hence (5) reduces to

$$\mathbf{a} = (\kappa \mathbf{n})v^2 + \mathbf{t}\,\frac{dv}{dt} = (\kappa v^2)\,\mathbf{n} + \frac{dv}{dt}\,\mathbf{t}$$

$$= \frac{v^2}{\rho}\,\mathbf{n} + \frac{dv}{dt}\,\mathbf{t} \qquad \cdots \quad (6)$$

Hence

The acceleration vector **a** *of a moving point P is a vector lying in the plane of the tangent and principal normal to the path of P ; that is, the acceleration vector* **a** *of the point P lies on the osculating plane. The tangential and normal components of* **a** *are*

$$\frac{dv}{dt} \text{ and } \kappa v^2 \left(= \frac{v^2}{\rho} \right).$$

The first of these two components gives the rate of increase of speed and is independent of the shape of the curve and the latter depends on the curvature as well as speed. We further recall here that curvature at a point of a curve has been defined as the arc-rate of rotation of the tangent at that point. Hence the normal component of acceleration may be written as

$$\kappa v^2 = \frac{d\theta}{ds} v^2 = \frac{d\theta}{ds}\,\frac{ds}{dt} v \qquad \left(\because\ \kappa = \frac{d\theta}{ds};\ v = \frac{ds}{dt} \right)$$

$$= v\,\frac{d\theta}{dt} = v\omega, \qquad \cdots \quad (7)$$

where ω is the time-rate of rotation of the tangent.

In particular, if a particle moves in a circle of radius r with *constant speed* v the tangential component of the acceleration is zero ($\because$ v is constant, $dv/dt = 0$) and the acceleration is then always normal to the path and equal to v^2/a ($\therefore$ $\rho = a$).

With rectangular cartesian coördinates

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

Hence :

The rectangular components of **v** *and* **a** *are the first and second derivatives of the coördinates.*

7·21. Radial and transverse components of velocity and acceleration.

We shall now consider the motion of a particle in a plane. The rectangular components of **v** and **a** at any point P of the path which is a plane curve given in terms of cartesian coördinates can be easily obtained by the rule given at the end of the previous article.

We next proceed to obtain the radial and cross-radial components of velocity and acceleration at a point P of the trajectory given by the polar equation

$$r = f(\theta),$$

which according to our notations of art. 6·31 (see also Fig. 6.4) can be written as

$$\mathbf{r} = r\mathbf{R}(\theta),$$

R being the unit vector along the radius vector **r**.

If we differentiate the above relation with respect to the time-variable t we get the velocity vector **v**. Thus if **P** be the unit vector in a direction perpendicular to **R**, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{R}) = \frac{dr}{dt}\mathbf{R} + r\frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P}.$$

$$\left(\because \quad \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{d\theta}\frac{d\theta}{dt} = \mathbf{P}\frac{d\theta}{dt}\right)$$

i.e., the components of **v** are

$$\frac{dr}{dt} \text{ and } r\frac{d\theta}{dt}$$

along **R** and **P** respectively.

Hence we find that the radial and transverse components of velocity are respectively.

$$\frac{dr}{dt} \text{ (or } \dot{r}\text{) and } r\,\frac{d\theta}{dt} \text{ (or } r\dot{\theta}\text{)}.$$

Now differentiating the expression for $d\mathbf{r}/dt$ once more we get the acceleration components. Thus,

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P}\right) \\
&= \left(\frac{d^2r}{dt^2}\mathbf{R} + \frac{dr}{dt}\frac{d\mathbf{R}}{dt}\right) + \left(\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{P} + r\frac{d^2\theta}{dt^2}\mathbf{P} + r\frac{d\theta}{dt}\frac{d\mathbf{P}}{dt}\right) \\
&= \left[\frac{d^2r}{dt^2}\mathbf{R} + \frac{dr}{dt}\frac{d\theta}{dt}\frac{d\mathbf{R}}{d\theta}\right] + \left[\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{P} + r\frac{d^2\theta}{dt^2}\mathbf{P} + r\left(\frac{d\theta}{dt}\right)^2\frac{d\mathbf{P}}{d\theta}\right] \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{R} + \left[2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right]\mathbf{P} \quad \left(\because \frac{d\mathbf{P}}{d\theta} = -\mathbf{R}\right)
\end{aligned}$$

The radial and transverse components of the acceleration are then the coefficients of **R** and **P** respectively.

For future references we collect the above results in the following forms (dots denote derivations with respect to t).

Velocity.

Radial : $\quad v_r = \dfrac{dr}{dt} = \dot{r}$

Transverse : $\quad v_p = r\dfrac{d\theta}{dt} = r\dot{\theta}$

Acceleration

Radial : $\quad a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$

Transverse : $\quad a_p = 2\,\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\dfrac{d^2\theta}{dt^2} = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

An important case arises in applications where the acceleration is always directed towards the origin *i.e.* acceleration is purely radial. Then

$$a_p = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

which gives $r^2\dfrac{d\theta}{dt} = h$ (constant).

This implies that the radius vector sweeps out area at a constant rate (*i.e.*, areal velocity is constant). For, the sectorial area measured from the initial line is

$$A = \tfrac{1}{2}\int_0^\theta r^2\,d\theta, \qquad \frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}h.$$

The expression for A used above refers to polar coördinates. The more general expression for dA/dt will be obtained in the next article.

7·3. Areal Velocity.

DEFINITION. The *areal velocity* of a point P about the point O, the origin of the reference system, is the time rate of description of the vector area swept out by the line OP.

Consider the general case of motion of a point P when its path is not necessarily a plane curve.

Let P, Q be the positions of the moving particle at times t, $t+\triangle t$ and $\mathbf{r}$ and $\mathbf{r}+\triangle\mathbf{r}$ be their respective position vectors relative to the origin O. (See Fig. 7.1) If $\triangle\mathbf{A}$ denote the vector area of the triangle OPQ, we know,

$$\begin{aligned}\triangle\mathbf{A} &= \tfrac{1}{2}\mathbf{r}\times(\mathbf{r}+\triangle\mathbf{r})\\ &= \tfrac{1}{2}\mathbf{r}\times\triangle\mathbf{r} \qquad (\because\ \mathbf{r}\times\mathbf{r}=\mathbf{0})\end{aligned}$$

so that

$$\frac{\triangle\mathbf{A}}{\triangle t} = \tfrac{1}{2}\mathbf{r}\times\frac{\triangle\mathbf{r}}{\triangle t}$$

Proceeding to the limit,

$$\frac{d\mathbf{A}}{dt} = \tfrac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt}.$$

$\therefore$ Areal velocity $= \tfrac{1}{2}\mathbf{r} \times \mathbf{v}$

The moment about O of the velocity vector $\mathbf{v}$ considered localised along the tangent line P is, by definition, $\mathbf{r} \times \mathbf{v}$, its direction being at right angles to the plane of $\mathbf{r}$ and $\mathbf{v}$. Now if p is the length of the perpendicular ON to the tangent at P, we have

$$\mathbf{r} \times \mathbf{v} = pv\,\mathbf{n} \quad \cdots \quad (1)$$

where v is the speed of P and $\mathbf{n}$ is the unit vector in the direction of normal to the plane of $\mathbf{r}$ and $\mathbf{v}$.

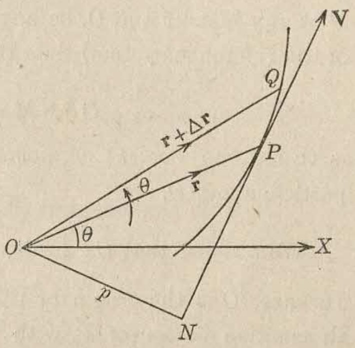


Fig. 7.1. Areal velocity

Thus $2 \times$ areal velocity
$=$ moment of $\mathbf{v}$ about 0

$$= pv\,\mathbf{n} \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

If now the particle moves in the plane OPN, we have, by using notations of the last article,

$$\begin{aligned}\mathbf{r} \times \mathbf{v} &= \mathbf{r} \times (\dot{r}\mathbf{R} + r\dot{\theta}\mathbf{P}) \\ &= \dot{r}(\mathbf{r} \times \mathbf{R}) + r\dot{\theta}(\mathbf{r} \times \mathbf{P}) \\ &= r\dot{\theta}(r\mathbf{n}) \qquad (\because\ \mathbf{r} \times \mathbf{R} = \mathbf{0},\ \mathbf{r} \times \mathbf{P} = \mathbf{n}) \\ &= r^2\dot{\theta}\,\mathbf{n} \qquad \cdots \quad (3)\end{aligned}$$

whence, $pv = r^2\dot{\theta} = r^2\omega$, where $\dot{\theta} = \omega$ is rate of turning of OP.

Hence, if we denote the magnitude of the areal velocity by $h/2$ then

$$h = |\mathbf{r} \times \mathbf{v}| = pv = r^2\frac{d\theta}{dt} = r^2\omega,$$

which is a very useful result.

7·31. Momentum : Moment of Momentum.

DEFINITION 1. By *momentum* $\mathbf{M}$ of a moving particle at any time we mean the vector $m\mathbf{v}$ where m is the mass and $\mathbf{v}$ is the velocity vector of the particle. Thus we write

$$\mathbf{M}=m\mathbf{v}.$$

DEFINITION 2. If $\mathbf{M}$ be the momentum vector of a particle P at any instant and O, be any point (not necessarily the origin of the reference system) then the vector

$$\overrightarrow{OP}\times\mathbf{M}=\mathbf{H}$$

is called the *moment of momentum* or *angular momentum* of the particle about O.

We observe that $\overrightarrow{OP}$ is the position vector of P relative to O. In case, O is the origin of the reference system then $\overrightarrow{OP}=\mathbf{r}$, is th position vector of P with reference to the origin and the angular momentum is given by

$$\mathbf{r}\times m\frac{d\mathbf{r}}{dt}=m\mathbf{r}\times\frac{d\mathbf{r}}{dt}$$

It needs hardly be stressed that the angular momentum is different for different positions of O.

Important Note.

We have remarked that the velocity and acceleration are relative concepts, relative to some reference system. We add here that the momentum $\mathbf{M}$ (also called *linear momentum*) is also a relative concept having meaning with reference to some given system of reference.

The angular momentum, on the other hand, is relative in two ways—(1) It is relative to the frame of reference with respect to which momentum is considered and (2) the point with respect to which the moment of the momentum is taken.

Rate of change of Linear Momentum.

The rate of change of linear momentum $\mathbf{M}$ is also a vector quantity given by

$$\frac{d\mathbf{M}}{dt}=\frac{d}{dt}(m\mathbf{v})=m\frac{d\mathbf{v}}{dt}+\mathbf{v}\frac{dm}{dt}$$

In Newtonian Dynamics, the mass m of the particle is supposed to be constant. Then, we get,

$$\frac{d\mathbf{M}}{dt}=m\frac{d\mathbf{v}}{dt}=m\mathbf{a}.$$

Thus the direction of $d\mathbf{M}/dt$ is the same as that of the acceleration vector $\mathbf{a}$.

7·4. Kinetics : Newton's Second Law of Motion.

The fundamental law of Newtonian dynamics is, in fact, the second law of Newton, which states :

The time rate of change of momentum of a particle is proportional to the impressed force and takes place in the direction in which the force acts.

Thus if $\mathbf{P}$ is the impressed force, this law states

$$\mathbf{P} \propto \frac{d\mathbf{M}}{dt}, \text{ where } \mathbf{M}=m\mathbf{v} \text{ ;}$$

where k is a scalar which gives the constant of proportionality. Assuming that the mass m remains constant, we get

$$\mathbf{P}=km\frac{d\mathbf{v}}{dt}=km\mathbf{a}.$$

Next we choose the unit of force as one which produces in a unit mass unit acceleration. Then we have $k=1$ and hence

$$\therefore \quad \mathbf{P}=m\mathbf{a}=m\frac{d\mathbf{v}}{dt}$$

This shows that the acceleration produced in the motion of a particle of constant mass has the same direction as the

force producing it. The above equation is known as the *equation of motion for a moving particle.*

7·41. Fundamental axiom : Equation of Motion.

The second Law of Newton implies much more than is asserted in the vector equation,

$$\mathbf{P} = m\mathbf{a}.$$

It implies that if a particle is in motion and several forces act upon it either in the same direction or in different directions then each force will produce its own contribution to the acceleration of the particle and this contribution is the same in magnitude and direction as it would be if the force considered were a *single force* acting upon the particle (*The Principle of the Physical Independence of Forces*).

Suppose the forces. $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$, when act separately on a particle produce accelerations $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ respectively. According to the above principle their combined effect when these forces act simultaneously on a particle is the same as that of a single force $\Sigma\mathbf{F}$ (*i.e*, the vector sum of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$) and it will produce a single acceleration $\mathbf{a}$ which is the vector sum of the accelerations $\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n$ so that we may remember,

$$m\mathbf{a} = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$$
$$= m(\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n).$$

In other words, the separate forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ are components of the single force $\Sigma\mathbf{F}$ and the separate accelerations $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are components of the single acceleration $\mathbf{a}$.

That is to say, the effect of each force is not influenced by other forces. Every individual force produces the same effect, as if, it were the only force acting on the particle.

This is the *Fundamental axiom* on which the structure of the Dynamical Theory is based and its justification is to be

found in the general agreement of the principle with the results of the everday experience and observation.

Let $\mathbf{r}$ be the position vector of the moving particle at any instant t (relative to a fixed point O) and suppose that with reference to a system of rectangular axes through O,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z) ;$$

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = (X, Y, Z).$$

Then the equation of motion $m\mathbf{a} = \Sigma\mathbf{F}$ gives

$$m\left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}\right) = \Sigma X\,\mathbf{i} + \Sigma Y\,\mathbf{j} + \Sigma Z\,\mathbf{k}$$

which gives three scalar equations :

$$m\ddot{x} = \Sigma X \,;\; m\ddot{y} = \Sigma Y \,;\; m\ddot{z} = \Sigma Z$$

These are the *Cartesian equations of motion* for the particle.

7·42. Principle of Angular Momentum.

Theorem. *The rate of increase of the angular momentum of a particle about O is equal to the moment about O of the resultant force acting on the particle.*

This result is known as the Principle of Angular Momentum.

Proof. Let $\mathbf{r}$ be the position vector of the moving particle relative to a fixed point O. Its velocity $\mathbf{v}$ is $\dot{\mathbf{r}}$ and its linear momentum is $\mathbf{M} = m\mathbf{v}$. Then the rate of increase of angular momentum $\mathbf{r} \times \mathbf{M}$ about O is

$$\begin{aligned}\frac{d}{dt}(\mathbf{r} \times \mathbf{M}) &= \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) \\ &= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} \\ &= \mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \mathbf{r} \times \mathbf{F},\end{aligned}$$

where $\mathbf{F}$ is the resultant force acting on the particle.

Hence the proposition.

In particular, if the force $\mathbf{F}$ has zero moment about O, the angular momentum of the particle about that point remains constant. This is the *Principle of conservation of angular momentum of the particle.*

The principle of angular momentum can be easily extended to a system of n particles having masses $m_1, m_2, \ldots, m_n$ with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ having impressed forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$. In this case the total angular momentum

$$\mathbf{H} = \sum_{k=1}^{n} m_k \, \mathbf{r}_k \times \mathbf{v}_k$$

and the total moment of the forces about O

$$= \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{F}_k = \sum_{k=1}^{n} \mathbf{r} \times \frac{d}{dt}(m_k \, \mathbf{v}_k) = \frac{d\mathbf{H}}{dt}.$$

Examples. VII(A)

1. A particle of mass 3 units has the position vector $\mathbf{r}$ at time t, referred to a fixed origin O and rectangular axes along $\mathbf{i}, \mathbf{j}, \mathbf{k}$ through O given by

$$\mathbf{r} = (\tfrac{1}{2}t^2 - 2t)\mathbf{i} + (\tfrac{1}{2}t^2 + 1)\mathbf{j} + \tfrac{1}{2}t^2\mathbf{k}.$$

Prove that the resultant force $\mathbf{F}$ acting on the particle is

$$\mathbf{F} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}.$$

2. In the previous example, evaluate for the particle at time t.

(i) its angular momentum about the origin O ;

(ii) time rate of change of angular momentum about O ;

(iii) the moment of $\mathbf{F}$ about the origin.

Verify the truth of the principle of angular momentum.

3. The position vector $\mathbf{r}$ with reference to a set of rectangular axes through O of a particle of mass m at time t is given by

$$\mathbf{r} = [a \cos nt,\ a \sin nt,\ \tfrac{1}{2}at^2].$$

Find the resultant force acting on the particle and the moment of the resultant force about O and then show that it is the same as the rate of increase of angular momentum of the particle about O.

4. A particle moves along the curve

$$\mathbf{r} = (t^3 - 4t)\,\mathbf{i} + (t^2 + 4t)\,\mathbf{j} + (8t^2 - 3t^3)\mathbf{k}$$

where the parameter t denotes the time-variable. Find the magnitudes of the tangential and normal components of its acceleration when $t = 2$.

5. Prove that the acceleration vector of a particle moving along a space curve lies on the osculating plane.

6. Prove that the radius of curvature ρ on a space curve is given by

$$\rho = \frac{v^3}{|\mathbf{v} \times \mathbf{a}|}.$$

7. Find the velocity and acceleration of a particle which moves along the curve

$$x = 2 \sin 3t,\ y = 2 \cos 3t,\ z = 8t$$

at any time t. Find the magnitude of the velocity and acceleration.

8. A constant force $\mathbf{F} = (6,\ -3,\ 3)$ acts on a particle of mass 4 units. Find the kinetic energy of the particle after 2 units of time if the initial velocity be $\mathbf{u} = (4,\ 3,\ -1)$

[*Remember* : The scalar quantity $\frac{1}{2}m\mathbf{v}^2$ is the *K. E.* of a particle of mass m when its velocity is $\mathbf{v}$].

9. A particle P moves in a plane with a constant *angular speed* ω about O. If $\overrightarrow{PO}$ and $d\mathbf{a}/dt$ are parallel, then show that

$$\frac{d^2r}{dt^2}=\tfrac{1}{3}r\omega^2.$$

10. P is a point on the tangent at a variable point Q to a fixed circle of radius a. If $QP=r$ and if it makes an angle θ with a fixed tangent, show that the components of acceleration of P along and perpendicular to QP are

$$\ddot{r}-r\dot{\theta}^2+a\ddot{\theta} \text{ and } \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})+a\dot{\theta}^2.$$

11. Find the areal velocity of a particle which moves along the path

$$\mathbf{r}=a\cos wt\ \mathbf{i}+b\sin wt\ \mathbf{j}$$

where a, b, w are constants and t is the time-variable.

Hints and Answers

1. $\mathbf{r}=(\tfrac{1}{2}t^2-2t,\ \tfrac{1}{2}t^2+1,\ \tfrac{1}{2}t^2)$; $\dot{\mathbf{r}}=\mathbf{v}=(t-2,\ t,\ t)$

$\mathbf{a}=\ddot{\mathbf{r}}=(1,\ 1,\ 1)$ (that is, acceleration is constant).

$\therefore\ \mathbf{F}=m\mathbf{a}=3(1,\ 1,\ 1)=3\mathbf{i}+3\mathbf{j}+3\mathbf{k}.$

Note that the magnitude of the force $\mathbf{F}=\sqrt{(3^2+3^2+3^2)}=3\sqrt{3}$ units.

2. Momentum $=\mathbf{M}=m\mathbf{v}=3(t-2,\ t,\ t)$.

(i) angular momentum $\mathbf{H}=\mathbf{r}\times\mathbf{M}=(\tfrac{1}{2}t^2-2t,\ \tfrac{1}{2}t^2+1,\ \tfrac{1}{2}t^2)\times(3t-6,\ 3t,\ 3t)$

$$=(3t,\ 3t^2,\ -3t^2-3t-6)$$

$$=3(t,\ t^2,\ -t^2-t-2).$$

(ii) $\dfrac{d}{dt}(\mathbf{r}\times\mathbf{M})=3(1,\ 2t,\ -2t-1).$

(iii) moment of $\mathbf{F}$ about $O=\mathbf{r}\times\mathbf{F}=(\tfrac{1}{2}t^2-2t,\ \tfrac{1}{2}t^2+1,\ \tfrac{1}{2}t^2)\times(3,\ 3,\ 3)$

$$=3(1.\ 2t,\ -2t-1).$$

4. Tangential, 16 ; Normal, $2\sqrt{73}$.

7. $\mathbf{v}=6\cos 3t\,\mathbf{i}-6\sin 3t\,\mathbf{j}+8\mathbf{k}$; $\mathbf{a}=-18\sin 3t\,\mathbf{i}-18\cos 3t\,\mathbf{j}$

hence $|\mathbf{v}|=10$; $|\mathbf{a}|=18$.

8. 52 units.

9. In this case $\mathbf{a}=(\ddot{r}-r\dot{\theta}^2)\,\mathbf{R}+(r\ddot{\theta}+2\dot{r}\dot{\theta})\,\mathbf{P}$ reduces to

$\mathbf{a}=(\ddot{r}-r\omega^2)\,\mathbf{R}+2\dot{r}\omega\,\mathbf{P}$ (for $\omega=\dot{\theta}$ is constant). Since $d\mathbf{a}/dt$ is parallel to $\overrightarrow{PO}$ $(-\mathbf{R})$, $\mathbf{P}$-component of $\dot{\mathbf{a}}$ is zero. This $\mathbf{P}$-component of $\dot{\mathbf{a}}$ can be shown to be $3\ddot{r}\omega-r\omega^3$. Equating this to zero we have the required result.

11. $\frac{1}{2}ab\omega\,\mathbf{k}$; remember areal velocity $=\frac{1}{2}\mathbf{r}\times\mathbf{v}$.

7·5. A Few common Problems of Particle Dynamics.

1. Motion under gravity :

If a moving particle of mass m be subjected to the action of gravity alone the equation of motion of the particle is

$$m\frac{d^2\mathbf{r}}{dt^2}=-mg\,\mathbf{k},$$

where $\mathbf{k}$ denotes the unit vector drawn vertically upwards. Cancelling m from both sides and integrating we get

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-gt\,\mathbf{k}+\mathbf{b} \qquad \cdots \quad (1)$$

where $\mathbf{b}$ is the constant vector of integration.

If $\mathbf{v}=\mathbf{u}$ when $t=0$ then $\mathbf{b}=\mathbf{u}$ from (1). Thus

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{u}-gt\,\mathbf{k} \qquad \cdots \quad (2)$$

Integrating again, we have

$$\mathbf{r}=\mathbf{u}t-\tfrac{1}{2}gt^2\mathbf{k}+\mathbf{c},$$

where $\mathbf{c}$ is the constant vector of integration.

Choosing $\mathbf{r}=\mathbf{0}$ when $t=0$ we get $\mathbf{c}=\mathbf{0}$. Hence

$$\mathbf{r}=\mathbf{u}t-\tfrac{1}{2}gt^2\mathbf{k}. \qquad \cdots \quad (3)$$

Thus the locus of $\mathbf{r}$ is a plane curve on the plane determined by the origin and the two vectors $\mathbf{u}$ and $\mathbf{k}$.

Considering the directions of $\mathbf{u}$ and $\mathbf{k}$ as x-axis and y-axis respectively, we find from (3) that the coördinates (x, y) of the moving point at time t are given by

$$x = |\mathbf{u}|t \; ; \quad y = -\tfrac{1}{2}gt^2$$

Elimination of t gives $y = -\frac{1}{2}\frac{g}{|\mathbf{u}|^2}x^2$.

This shows that the path is a parabola.

Hence the locus of a particle moving freely in a constant gravitational field is a parabola.

In case $\mathbf{u}$ has the direction of $\mathbf{k}$ *i.e.*, velocity at time $t=0$ is vertical, we conclude from (3) that the particle is moving in the vertical line through the position of the particle at time $t=0$.

In the general case when $\mathbf{u}$ and $\mathbf{k}$ have different directions, say the particle is projected with a velocity u making an angle α with the direction of x-axis then we may put

$$\mathbf{u} = u \cos \alpha \; \hat{\mathbf{u}} + u \sin \alpha \; \mathbf{k}$$

($\hat{\mathbf{u}}$ is the unit vector along $\mathbf{u}$).

Now (3) gives

$$x = u \cos \alpha \; t, \; y = u \sin \alpha \; t - \tfrac{1}{2}gt^2.$$

Since $y=0$ at time $T = (2u \sin \alpha)/g$, the range on the horizontal plane is

$$R = u \cos \alpha . T = (u^2 \sin 2\alpha)/g.$$

Again at the highest point, the velocity is horizontal and hence

$$\mathbf{v}.\mathbf{k} = 0 \; ; \; i.e., \; (\mathbf{u} - gt\mathbf{k}).\mathbf{k} = 0$$

so that $t = \mathbf{u}.\mathbf{k}/gk^2 = (u \sin \alpha)/g$ and hence the maximum height attained is obtained by putting the value of t in the expression for y; thus the maximum height attained is $u^2 \sin^2 \alpha/2g = H$ (say).

2. Motion under gravity subjected to resistance proportional to velocity.

In this case the equation of motion is given by

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{k} - m\mu\frac{d\mathbf{r}}{dt}\ (\mu = \text{constant of proportionality})$$

$$i.e.,\ \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - \mu\frac{d\mathbf{r}}{dt}.$$

On integration,

$$\frac{d\mathbf{r}}{dt} + \mu\mathbf{r} = -gt\,\mathbf{k} + \mathbf{b},$$

where $\mathbf{b}$ = constant vector of integration.

If $\mathbf{r} = \mathbf{0}, \frac{d\mathbf{r}}{dt} = \mathbf{u}$ when $t = 0$, we thus obtain $\mathbf{b} = \mathbf{u}$.

Thus,

$$\frac{d\mathbf{r}}{dt} + \mu\mathbf{r} = -gt\,\mathbf{k} + \mathbf{u} \qquad \cdots \quad (4)$$

$$\therefore \qquad e^{\mu t}\left[\frac{d\mathbf{r}}{dt} + \mu\mathbf{r}\right] = \left(\mathbf{u} - gt\,\mathbf{k}\right)e^{\mu t}.$$

Now, on integration

$$\mathbf{r}e^{\mu t} = \mathbf{u}\int e^{\mu t}dt - g\mathbf{k}\int te^{\mu t}dt$$

$$= \frac{\mathbf{u}}{\mu}e^{\mu t} - g\mathbf{k}\left[\frac{te^{\mu t}}{\mu} - \frac{e^{\mu t}}{\mu^2}\right] + \mathbf{c}.$$

Taking $\mathbf{r} = \mathbf{0}$, when $t = 0$, we obtain $\mathbf{c} = -\frac{\mathbf{u}}{\mu} - \frac{g\mathbf{k}}{\mu^2}$

$$\therefore \quad \mathbf{r} = \frac{1}{\mu}\left(1 - e^{-\mu t}\right)\mathbf{u} + \left[\frac{g}{\mu^2}\left(1 - e^{-\mu t}\right) - \frac{gt}{\mu}\right]\mathbf{k} \qquad \cdots \quad (5)$$

Then from (5) it follows that the locus is a plane curve whose plane passes through the origin and will lie in the plane of $\mathbf{u}$ and $\mathbf{k}$. From (5) it also follows that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^{-\mu t}\mathbf{u} + \frac{g}{\mu}\left(e^{-\mu t} - 1\right)\mathbf{k}$$

so that when $t \to \infty$, $\mathbf{v} \to -\frac{g}{\mu}\mathbf{k}$ which is known as the *limiting* velocity of the particle.

Again the particle will move horizontally at time t given by $\mathbf{v} . \mathbf{k} = \mathbf{0}$, which gives

$$t = \frac{1}{\mu} \log\left(1 + (\mu/g)\, \mathbf{u}.\, \mathbf{k}\right).$$

3. Harmonic Motion.

The equation of motion of a particle subjected to a force towards a fixed point and with magnitude proportional to its distance from the fixed point is given by

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu\mathbf{r} \; (\mu > o),$$

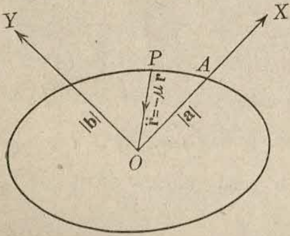


Fig. 7.2. Harmonic motion

the origin being taken as the fixed point.

The general solution of this equation is

$$\mathbf{r} = \mathbf{a} \cos \sqrt{\mu}t + \mathbf{b} \sin \sqrt{\mu}t, \quad \cdots \quad (6)$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

From (6) we also obtain,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\sqrt{\mu}\,\mathbf{a} \sin \sqrt{\mu}t + \sqrt{\mu}\,\mathbf{b} \cos \sqrt{\mu}t \quad \cdots \quad (7)$$

See that at $t = 0$, $\mathbf{r} = \mathbf{a}$, $\mathbf{v} = \sqrt{\mu}\,\mathbf{b}$. So if the initial position and velocity is given we can fix up $\mathbf{a}$ and $\mathbf{b}$.

If the directions of $\mathbf{a}$ and $\mathbf{b}$ are taken as the x-axis and y-axis respectively, we have from (6), for any point (x, y) on the curve,

$$x = |\mathbf{a}| \cos \sqrt{\mu}t$$

$$y = |\mathbf{b}| \sin \sqrt{\mu}t$$

whence $\dfrac{x^2}{|\mathbf{a}|^2} + \dfrac{y^2}{|\mathbf{b}|^2} = 1$

so that the path is an ellipse (Fig. 7.2) such that the diameters of the ellipse along **a** and **b** are conjugate.

From (6) and (7) we find that **r** and **v** are periodic functions of t; the period being $2\pi/\sqrt{\mu}$ which is the time required to describe the ellipse once.

7·51. Central forces : Inverse square law of attraction.

DEFINITION. If a particle be acted on by a force which is always directed towards a fixed point, we say that the particle is subjected to a *Central force*, the fixed point being the *Centre of force.* The exact law of force may be some function of the distance.

Suppose a particle be subjected under a central force, the origin O being the centre of force, the exact law of force as a function of distance of the particle from O being left indeterminate. We shall now prove :

The path described by the particle will be a plane curve and the areal velocity and angular momentum relative to the fixed point O will be constant.

This path is known as *central orbit.*

The differential equation of motion can be written as

$$\frac{d^2\mathbf{r}}{dt^2}=f(r)\,\hat{\mathbf{r}}$$

As the acceleration is always along (and therefore parallel to) the radius vector, the cross product

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{0},$$

which, on integration gives,

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=\text{constant vector } \mathbf{h} \text{ (say)} \qquad \cdots \quad (1)$$

Thus **h** is a vector perpendicular to the plane of **r** and $\frac{d\mathbf{r}}{dt}$.

The equation (1) also shows that the *areal velocity* as well as *angular momentum* about O is constant.

Taking scalar product of (1) by $\mathbf{r}$ we obtain

$$\mathbf{r} \cdot \mathbf{h} = 0 \qquad \cdots \quad (2)$$

so that the locus of the point $\mathbf{r}$ is *a curve on the plane perpendicular to the constant vector* $\mathbf{h}$.

1. Given a definite law of force, to find the orbit.

In the following we shall find the path described by a particle subjected to a central force (the centre of force is at the origin O) which obeys the inverse square law (*i.e.*, the force varies inversely as the square of the distance from O). Since the acceleration is directed towards O, its direction is given by $-\mathbf{r}$, where $\mathbf{r}$ is the position vector of the particle relative to O and hence the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^2}\hat{\mathbf{r}} \qquad \cdots \quad (3)$$

where $|\mathbf{r}| = r$ and $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$.

The constant μ, called the *intensity of force*, denotes the force on unit mass at unit distance from O.

We proceed to show that the path of $\mathbf{r}$ is a *conic*. There is no difficulty to follow that the relations (1) and (2) will be still true. From (3) we further obtain,

$$\frac{1}{\mu}\frac{d^2\mathbf{r}}{dt^2} \times \mathbf{h} = -\frac{1}{r^2}(\hat{\mathbf{r}} \times \mathbf{h}) = -\frac{1}{r^2}\left[\hat{\mathbf{r}} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right)\right] \qquad [\text{ use } (1)]$$

$$= -\frac{1}{r^2}\left[-(\hat{\mathbf{r}} \cdot \mathbf{r})\frac{d\mathbf{r}}{dt} + \left(\hat{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt}\right)\mathbf{r}\right]$$

$$= -\frac{1}{r^2}\left[-r\frac{d\mathbf{r}}{dt} + \frac{dr}{dt}\mathbf{r}\right]$$

[For by art. 7·21, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P}\ ;\ \text{ hence } \hat{\mathbf{r}}\,.\,\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\Big]$$

$$\therefore\ \frac{1}{\mu}\frac{d^2\mathbf{r}}{dt^2}\times\mathbf{h} = \frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right). \qquad \cdots\ (4)$$

Integrating (4), we get

$$\frac{1}{\mu}\frac{d\mathbf{r}}{dt}\times\mathbf{h} = \frac{\mathbf{r}}{r} + \mathbf{a} \qquad \cdots\ (5)$$

where $\mathbf{a}$ is an arbitrary constant vector. That $\mathbf{a}$ lies in the plane of the orbit follows from the fact that

$$\frac{1}{\mu}\frac{d\mathbf{r}}{dt}\times\mathbf{h}\,.\,\mathbf{h} = \frac{1}{r}\mathbf{r}\,.\,\mathbf{h} + \mathbf{a}\,.\,\mathbf{h}$$

whence $\quad \mathbf{a}\,.\,\mathbf{h} = 0 \quad$ [use (2)]

From (5), we also obtain,

$$\mathbf{r}\,.\left(\frac{1}{\mu}\frac{d\mathbf{r}}{dt}\times\mathbf{h}\right) = \frac{1}{r}\mathbf{r}\,.\,\mathbf{r} + \mathbf{r}\,.\,\mathbf{a}$$

$$\text{or}\quad \frac{1}{\mu}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\cdot\mathbf{h}\right) = \frac{1}{r}r^2 + |\,\mathbf{a}\,|\,r\cos\theta$$

where θ is the (variable) inclination of $\mathbf{r}$ to $\mathbf{a}$. Hence, using (5) we have

$$1/\mu\ \mathbf{h}\,.\,\mathbf{h} = r(1 + |\,\mathbf{a}\,|\cos\theta)$$

$$\text{or}\quad h^2/\mu = r(1 + e\cos\theta),\quad \text{if } e = |\,\mathbf{a}\,| \qquad \cdots\ (6)$$

We usually write $h^2/\mu = l$, then (6) reduces to

$$\frac{l}{r} = 1 + e\cos\theta \qquad \cdots\ (7)$$

representing a conic whose eccentricity is e and semi-latus-rectum is $l = h^2/\mu$ and one of the foci is at the centre of force.

The conic is a parabola, ellipse or hyperbola according as e is equal to, less than or greater then unity.

2. To find the value of eccentricity in terms of initial speed.

Squaring (5), we obtain

$$\mathbf{a}^2 = e^2 = \left(\frac{1}{\mu}\mathbf{v}\times\mathbf{h} - \frac{\mathbf{r}}{r}\right)^2 = \frac{h^2v^2}{\mu^2} + 1 - \frac{2}{\mu r}\,\mathbf{v}\times\mathbf{h}\,.\,\mathbf{r}$$

$$= \frac{h^2v^2}{\mu^2} + 1 - \frac{2h^2}{\mu r} \quad (\because \quad \mathbf{v}\times\mathbf{h}\,.\,\mathbf{r} = \mathbf{r}\times\mathbf{v}\,.\,\mathbf{h} = \mathbf{h}\,.\,\mathbf{h} = h^2)$$

It follows that

$$e^2 <, = \text{ or } > 1 \text{ according as } v^2 <, = \text{ or } > \frac{2\mu}{r}.$$

If now a particle be projected with a *speed* v from a distance c from the centre of force, the path is an ellipse, parabola or hyperbola according as

$$v^2 < = \text{ or } > \frac{2\mu}{c}.$$

The eccentricity in all cases is given by

$$e = \sqrt{\frac{h^2}{\mu}\left(\frac{v^2}{\mu} - \frac{2}{c}\right) + 1} \qquad \cdots \quad (8)$$

3. To find the speed v at any point of the orbit.

Multiplying (3) scalarly by $2\frac{d\mathbf{r}}{dt}$ we obtain

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = \frac{-2\mu}{r^2}\hat{\mathbf{r}}\cdot\frac{d\mathbf{r}}{dt} = \frac{-2\mu}{r^2}\frac{dr}{dt}.$$

On integration, we get

$$v^2 = \left(\frac{d\mathbf{r}}{dt}\right)^2 = C + \frac{2\mu}{r} \qquad \cdots \quad (9)$$

If $v = v_0$ when $r = a$ *i.e.* if the speed at some point be given then we obtain

$$C = v_0{}^2 - \frac{2\mu}{a}$$

whence, $v^2 = v_0{}^2 + 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)$ $\qquad \cdots \quad (10)$

which gives speed at any point of the path in terms of the speed at some assigned point.

4. Planetary motion.

The most important case arises when the orbit is an ellipse. The motions of planets relative to the sun are of this description.

If the elliptic orbit has the two semi-axes, a, b then its area is πab and hence the *periodic time*

$$T=\frac{\pi ab}{h/2}=\frac{2\pi\, ab}{\sqrt{\mu l}}=\frac{2\pi\, ab}{\sqrt{\mu b^2/a}}=\frac{2\pi}{\sqrt{\mu}}\,a^{\frac{3}{2}}\,.$$

In other words, the square of the periodic time is proportional to the cubes of the semi-major axes of the elliptic orbit. This relation was observed by Kepler in the case of planets.

In order to determine C of (9) we consider the speed v_0 at the end of the minor axis. The perpendicular distance to the tangent at this point is b, while the distance r from the focus is a. Hence (9) gives

$$C+\frac{2\mu}{a}=v_0{}^2=\frac{h^2}{b^2}=\mu\frac{l}{b^2}=\frac{\mu}{a}\,.$$

whence, $C=-\mu/a$ and the speed v at any point is given by

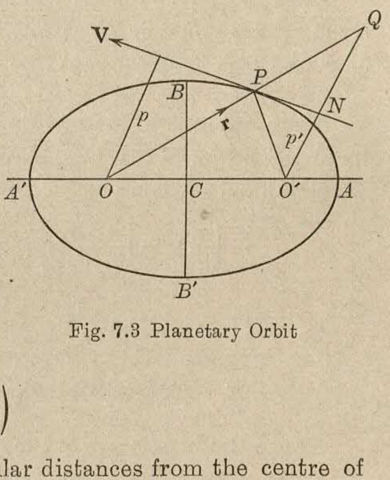


Fig. 7.3 Planetary Orbit

$$v^2=2\mu\left(\frac{1}{r}-\frac{1}{a}\right)$$

If p and p' are the perpendicular distances from the centre of force O (which is one of the foci) and the other focus O' respectively to the tangent to the ellipse at P (Fig. 7.3), the speed v at P is given by

$$v=h/p=hp'/pp'=hp'/b^2.$$

5. Hodograph.

DEFINITION. The hodograph of a point P moving along a curve is the locus of a second point Q whose position vector at

any instant is equal (or proportional) to the *velocity vector* of P at that instant.

Let a point P move under a constant acceleration $g\mathbf{d}$, where $\mathbf{d}$ is the unit vector in the direction of acceleration. Then, we have $\dot{\mathbf{v}} = g\mathbf{d}$ which, on integration, gives

$\mathbf{v} = \mathbf{v}_0 + gt\mathbf{d}$ ($\mathbf{v}_0$ = velocity at the instant $t = 0$).

The position vector $\mathbf{R}$ of Q is then equal (or proportional) to $\mathbf{v}_0 + gt\mathbf{d}$. The locus of Q is thus a straight line parallel to $\mathbf{d}$ and the velocity of Q *i.e.*, $\dot{\mathbf{R}}$ is equal or proportional to $g\mathbf{d}$, that is to the acceleration of P.

Thus the velocity with which Q describes the hodograph has a magnitude equal or proportional to the magnitude of acceleration of P and they have the same direction.

Again suppose a particle describes a conic under a central force to the focus the hodograph will be a circle. For, by (5) we obtain, on squaring,

$$\left(\frac{\mathbf{r}}{r}\right)^2 = \left(\frac{1}{\mu}\frac{d\mathbf{r}}{dt} \times \mathbf{h} - \mathbf{a}\right)^2 = \left(\frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \mathbf{a}\right)^2$$

or $$1 = \frac{1}{\mu^2}h^2\mathbf{v}^2 - \frac{2}{\mu}\mathbf{v} \times \mathbf{h}.\mathbf{a} + \mathbf{a}^2$$

or $$\mathbf{v}^2 - 2\mathbf{v}.\mathbf{h} \times \mathbf{a}\frac{\mu}{h^2} + \frac{e^2\mu^2}{h^2} = \frac{\mu^2}{h^2} \quad (\text{assume } \mathbf{a} = e\hat{\mathbf{a}})$$

Hence the hodograph is the circle

$$\mathbf{r}^2 - 2\mathbf{r}.\mathbf{c} + (e^2 - 1)\frac{\mu^2}{h^2} = 0$$

whose centre is the point

$$\mathbf{c} = \frac{\mu}{h^2}\mathbf{h} \times \mathbf{a} = \frac{e\mu}{h}\hat{\mathbf{h}} \times \hat{\mathbf{a}} \quad (\because\ \mathbf{a} = e\hat{\mathbf{a}}).$$

When the conic is a parabola, the hodograph is

$$\mathbf{r}^2 - 2\mathbf{r}.\mathbf{c} = 0$$

i e. it passes through the origin.

Examples. VII(B)

1. A particle P describes an elliptic orbit under a central force to the focus O (Fig. 7.3). Show that its velocity at any instant may be resolved into two component velocities of constant magnitudes, perpendicular to the major axis and OP respectively.

Solve this problem in two ways :

(i) using speed $v \propto p'$.

(ii) taking vector product of equation (5) of art. 7·51 with $\mathbf{h}$.

2. Three particles A, B, C whose position vectors at some instant are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ have velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ respectively. Prove that the rate of change of vector area of the triangle ABC is

$$\tfrac{1}{2}\{\mathbf{r}_1 \times (\mathbf{v}_2 - \mathbf{v}_3) + \mathbf{r}_2 \times (\mathbf{v}_3 - \mathbf{v}_1) + \mathbf{r}_3 \times (\mathbf{v}_1 - \mathbf{v}_2)\}.$$

3. Show that if $\mathbf{a}$ is a fixed radius of a circle and $\mathbf{k}$ is a unit vector drawn from the centre of the circle at right angles to its plane, the equation of the circle is

$$\mathbf{r} = \mathbf{a} \cos \theta + (\mathbf{k} \times \mathbf{a}) \sin \theta,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{a}$.

Deduce from the above example that the velocity in the circle is $(\mathbf{k} \times \mathbf{r})\, d\theta/dt$ and that its moment about the centre is the vector

$$a^2 \frac{d\theta}{dt} \mathbf{k}.$$

Also obtain the acceleration in the circle in the form

$$\ddot{\mathbf{r}} = (\mathbf{k} \times \mathbf{r}) \frac{d^2\theta}{dt^2} - \mathbf{r} \left(\frac{d\theta}{dt}\right)^2.$$

4. Show that the equation of an ellipse with a focus as origin may be written as

$$\mathbf{r} = (\mathbf{l} \sin \theta + \mathbf{l} \times \mathbf{k} \cos \theta)/(1 + e \cos \theta),$$

where e is the eccentricity, $\mathbf{l}$ a vector represented by semi-latus rectum and $\mathbf{k}$ a unit vector perpendicular to the plane. Find an expression for the velocity in the ellipse and verify that its moment about the focus is $\mathbf{r}\times\dot{\mathbf{r}}=r^2\dot{\theta}\,\mathbf{k}$. Assuming this to be a constant h, show that the acceleration is h^2/lr^2 directed towards the focus.

5. A particle moves under a central force obeying the law that it is proportional to the cube of the distance r from the centre of force. Show that

$$r^2 = at^2 + \beta,$$

where a and β are constants.

6. A particle describes an elliptic path under a central force to the centre of the ellipse. Show that its angular speed about a focus varies inversely as its distance from that focus and that the sum of the reciprocals of its angular speeds about the two foci is constant.

7. If a particle describes an ellipse with the centre of force as one of the foci then show that the speed at the end of the minor axis is a mean proportional between the speeds at the ends of any diameter and that the angular speed about the other focus varies inversely as the square of the normal.

8. Discuss the motion of a particle under a central force, the law of force being any function of the distance. [See hints]

9. Use Example 8 to find the law of force when the path is given to be an ellipse

$$\frac{1}{p^2}=\frac{a}{b^2}\left(\frac{2}{r}-\frac{1}{a}\right)$$

10. Find the law of force to the *centre* of an ellipse under which the ellipse will be described. Remember that for an ellipse with origin at the centre we have

$$\frac{1}{p^2}=\frac{a^2+b^2-r^2}{a^2b^2}.$$

11. Find the law of force under which the following curves are described :

(i) Equiangular spiral (centre of force is the pole)

(ii) Circle with the centre of force on the circumference How do the speeds vary in the two cases ?

Hints and Answers

1. (i) See Fig. 7.3. $\mathbf{v}$ is proportional to $\overrightarrow{O'N}$ and at *right angles* to it, But if $O'N$ is produced to meet OP in Q then

$$\overrightarrow{O'N}=\tfrac{1}{2}\overrightarrow{O'Q}=\tfrac{1}{2}(\overrightarrow{O'O}+\overrightarrow{OQ}),$$

and both $O'O$ and OQ are of constant length ; hence etc.

3. If $\hat{\mathbf{a}}$ be the unit vector along $\mathbf{a}$ then $\mathbf{k}\times\hat{\mathbf{a}}$ is a unit vector in the plane of the circle perpendicular to the direction of $\mathbf{a}$. If a be the radius of the circle then the position vector $\mathbf{r}$ of any point on the circle is given by

$$\mathbf{r}=a\cos\theta\,\hat{\mathbf{a}}+a\sin\theta(\mathbf{k}\times\hat{\mathbf{a}})=\mathbf{a}\cos\theta+(\mathbf{k}\times\mathbf{a})\sin\theta.$$

Hence obtain $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ (θ is the scalar variable).

8. The equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2}=-f(r)\frac{\mathbf{r}}{r}.$$

Form scalar product of each side with $2\dfrac{d\mathbf{r}}{dt}$ and obtain

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}=-2f(r)\frac{\mathbf{r}}{r}\cdot\frac{d\mathbf{r}}{dt}=-2f(r)\frac{dr}{dt}.$$

On integration we get

$$v^2=\left(\frac{d\mathbf{r}}{dt}\right)^2=C-2\int f(r)\,dr$$

This gives the speed at any distance. Writing h^2/p^2 for v^2 and differentiating the last equation with respect to r ; we find

$$\frac{h^2}{p^3}\frac{dp}{dr}=f(r),$$

which gives the pedal equation of the path, $f(r)$ being given.

11. (i) $f(r)=\mu/r^2$; $v\propto 1/r$; (ii) $f(r)=\mu/r^5$, $v\propto 1/r^2$.

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CALCUTTA UNIVERSITY QUESTIONS

B. A. and B. Sc. Examinations

1960

Honours Course (*New Syllabus*)

Answer *any two* questions.

1. If the position vectors of three points be denoted by α, β, γ show that the necessary and sufficient condition that the points be collinear is that there exists a linear relation $p\alpha + q\beta + r\gamma = 0$, in which p, q, r are numbers which satisfy $p + q + r = 0$.

Apply vector method to show that the internal bisector of the angle A of a triangle ABC divides the side BC in the ratio $AB : AC$.

If s be the length of an arc measured from a fixed point on a given curve up to a variable point P whose position vector is α, find the vector equation of the tangent to the curve at P. What are the direction cosines of the tangent ?

2. Define dot product of vectors and show that it satisfies the distributive law [*i.e.* $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$]. Use this property to show that the work done by a number of forces acting on a particle is the same as that done by their resultant.

If α is a constant vector and β is a variable vector drawn from the same origin and c is a number, show that the equation of a plane can be put in the form $\alpha.\beta = c$; conversely, any equation of this form represents a plane. Show also that the equation of a sphere can be put as $\beta^2 - 2\alpha.\beta + k = 0$, where k is a number. Deduce that the diameter of a sphere subtends a right angle at any point of the surface.

3. Define cross-product of two vectors. How does the product behave regarding the commutative law? Give geometrical interpretation.

With reference to an orthogonal coördinate system, show that

$$\alpha \times \beta = (a_2 b_3 - a_3 b_2)\mathbf{x} + (a_3 b_1 - a_1 b_3)\mathbf{y} + (a_1 b_2 - a_2 b_1)\mathbf{z}$$

where **x, y, z** are unit vectors in the positive directions of the axes, and the a's and b's are components of the vectors α and β.

Denoting $\alpha.(\beta \times \gamma) = [\alpha\beta\gamma]$, interpret $[\alpha\beta\gamma] = 0$.

Show that any four vectors satisfy

$$[\alpha\beta\gamma]\delta = [\beta\gamma\delta]\alpha + [\gamma\alpha\delta]\beta + [\alpha\beta\delta]\gamma.$$

Pass Course (*New Syllabus*)

Answer *any one* question.

1. (*a*) Find the condition that the extremities of three coinitial coplanar vectors are collinear.

(*b*) If the external bisectors of the angles of a triangle intersect the opposite sides in the points P, Q, R, prove by the method of vectors that P, Q, R are collinear.

2. (*a*) Define vector product $\alpha \times \beta$. Discuss the significance of the vector triple product $\alpha \times (\beta \times \gamma)$.

(*b*) Prove that $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ if and only if $(\gamma \times \alpha) \times \beta = 0$.

Honours Course (*Old Syllabus*)

1. Obtain the vector equation of a straight line joining the points whose position vectors are **a** and **b** referred to a fixed origin.

Forces P, Q act at O and have a resultant R. If any transversal cuts their lines of action at A, B, C respectively, show, by vector methods

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}.$$

Prove that the joins of the mid-points of opposite edges of a tetrahedron intersect and bisect each other.

2. Define the scalar and vector triple products of three given vectors and interpret them geometrically.

If **a, b, c** are any three non-coplanar vectors, show that any vector **r** may be expressed as

$$\mathbf{r}=\frac{[\mathbf{r\ b\ c}]\mathbf{a}+[\mathbf{r\ c\ a}]\,\mathbf{b}+[\mathbf{r\ a\ b}]\,\mathbf{c}}{[\mathbf{a\ b\ c}]}.$$

3. Show that the equation of the plane containing the two parallel lines

$$\mathbf{r}=\mathbf{a}+s\mathbf{b},\ \mathbf{r}=\mathbf{a}'+t\mathbf{b}$$

is $$\mathbf{r}.\,(\mathbf{a}'-\mathbf{a})\times\mathbf{b}=[\mathbf{a\ a}'\ \mathbf{b}].$$

Find the locus of a point which is equidistant from the three planes

$$\mathbf{r}.\,\mathbf{n}_1=q_1,\quad \mathbf{r}\,.\,\mathbf{n}_2=q_2,\quad \mathbf{r}\,.\,\mathbf{n}_3=q_3$$

1961

Honours Course

Answer *any two* questions.

1. (*a*) If the position vectors of four points, no three of which are collinear, be denoted by α, β, γ, δ show that a necessary and sufficient condition that the points be coplanar is that there exist four numbers p, q, r, s, not all zero, such that the relations $p\alpha+q\beta+q\gamma+s\delta=0$ and $p+q+r+s=0$ are satisfied.

Show that the following four points are coplanar :

$$\lambda=6\alpha-4\beta+10\gamma,\ \mu=-5\alpha+3\beta-10\gamma,\ \nu=4\alpha-6\beta-10\gamma,\ \rho=2\beta+10\gamma.$$

(*b*) Show that the centroid of n given points with associated masses is independent of the origin of the position vectors of the points.

If the position vectors of n points represent n concurrent forces, show that the forces will be in equilibrium if the centroid of the points coincides with the origin.

2. (*a*) Define dot product of two vectors. Apply the definition to obtain the trigonometrical formula $c = b \cos A + a \cos B$ for a triangle ABC.

(*b*) Obtain the equation of a plane in the form $\lambda.\eta = p$, where η denotes the unit vector perpendicular to the plane.

Find the point where this plane meets the line $\lambda = \alpha + t\beta$.

(*c*) Define vector product of two vectors.

Show that $\alpha.(\beta \times \gamma)$ is equal in magnitude to the volume of the parallelopiped of which α, β, γ are coterminous edges.

3. (*a*) Prove the following relations :

(*i*) $\alpha \times (\beta \times \gamma) = (\alpha.\gamma)\ \beta - (\alpha.\beta)\gamma$,

(*ii*) $(\beta + \gamma).\{(\gamma + \alpha) \times (\alpha + \beta)\} = 2[\alpha\beta\gamma]$.

(*b*) A particle of mass 5 units has the position vector λ at a t given by $\lambda = (\frac{1}{2}t^2 - 2t)\ \mathbf{X} + (t^2 + 1)\ \mathbf{Y} + \frac{3}{2}t^2\mathbf{Z}$, where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are unit vectors in the positive directions of the axes.

Show that the resultant force acting on the particle is

$$5\mathbf{X} + 10\mathbf{Y} + 15\mathbf{Z}.$$

Pass Course

Answer *any one* question.

1. Show that the vectors

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},\ \mathbf{B} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k},\ \mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

form the sides of a right-angled triangle.

[**Hints.** First show that they form a triangle and then prove that the triangle is *right*]

2. (*a*) Define the scalar and vector products of two vectors and state their geometrical significance.

(*b*) Prove by vector methods that the medians of a triangle are concurrent.

1962

Honours Course

Answer *any two* questions.

1. (*a*) State and establish the distributive law for vector products.

(*b*) Show that the points $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $3(\mathbf{i}+\mathbf{j}+\mathbf{k})$ are equidistant from the plane $\mathbf{r}.(5\mathbf{i}+2\mathbf{j}-7\mathbf{k})+9=0$, where $\mathbf{r}$ is the position vector of any point on the plane and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the coördinate axes.

(*c*) A particle, acted on by constant forces $4\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$, is displaced from the point $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ to the point $5\mathbf{i}+4\mathbf{j}+\mathbf{k}$. Calculate the total work done by the forces ; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the coördinate axes.

2. (*a*) $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors, shew that

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b}-[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}.$$

Hence, or otherwise, shew that

$$[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]\mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c}=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d},$$

where $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ represents the scalar triple product of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(*b*) The velocity of a boat relative to water flowing in a river is represented by $3\mathbf{i}+4\mathbf{j}$ and that of the water relative to the ground by $\mathbf{i}-3\mathbf{j}$. Find the velocity of the boat relative to the ground if $\mathbf{i}$ and $\mathbf{j}$ represent velocities of one mile per hour towards East and North respectively.

3. (a) Obtain the vector equation to the straight line through the points $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $3\mathbf{k}-2\mathbf{j}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the axes of coördinates. Find also the position vector of the point where the above line cuts the plane through the origin containing the points $4\mathbf{j}$ and $2\mathbf{i}+\mathbf{k}$.

(*b*) A particle P moving on a plane has its coördinates (r, θ) at any time t; shew that if $\mathbf{v}$ and $\mathbf{f}$ are the velocity and acceleration of the particle ar time t, then

$$\mathbf{v}=\dot{r}\,\hat{\mathbf{r}}+r\,\omega\,\hat{\mathbf{s}}$$

and $$\mathbf{f}=(\ddot{r}-r\,\dot{\theta}^2)\,\hat{\mathbf{r}}+(r\,\ddot{\theta}+2\dot{r}\,\dot{\theta})\,\hat{\mathbf{s}}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$ are unit vectors parallel and perpendicular to the radius vector to P and ω is the angular speed of P about the origin of coördinates.

Pass Course

Answer *any one* question.

1. Prove by vector methods that for any triangle ABC

$$c^2=a^2+b^2-2ab\cos C.$$

Prove that if $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are vectors

$$\mathbf{P}\times(\mathbf{Q}\times\mathbf{R})=(\mathbf{P}.\,\mathbf{R})\,\mathbf{Q}-(\mathbf{P}.\,\mathbf{Q})\mathbf{R}.$$

2. (i) The position vectors of two points P and Q are respectively $3\mathbf{i}+7\mathbf{j}-4\mathbf{k}$ and $6\mathbf{i}-2\mathbf{j}+12\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the axes of coördinates. Calculate the angle between OP and OQ, where O is the origin.

(ii) If the vertices of a triangle are points $\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, what are the vectors determined by its sides? $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the axes of coördinates.

THREE-YEAR DEGREE COURSE

1963 (Pass)

1. (*a*) Find the position vector of the point which divides the join of two points with position vectors α and β in a given ratio

$$m : n \;(> 0).$$

(*b*) Show that $(\alpha \times \beta)^2 = \alpha^2\beta^2 - (\alpha \,.\, \beta)^2$ where α, β are any two vectors.

2. (*a*) If the diagonals of a quadrilateral bisect each other, show, by vector method, that the figure is a parallelogram.

(*b*) Show by vector method that the perpendicular bisectors of the sides of a triangle are concurrent.

1963 (Honours)

1. (*a*) Find the condition that the extremities of three coinitial coplanar vectors are collinear.

(*b*) From points in the base of a triangle, straight lines are drawn parallel to the sides. Show by the method of vectors that the intersection of the diagonals of each of the parallelograms so formed, lie on a straight line.

2. (*a*) Define the vector product $\alpha \times \beta$ of two vectors α and β and prove that this product satisfies the distributive law :

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma.$$

How are the three vector products $\alpha \times \beta$, $\beta \times \gamma$ and $\gamma \times \alpha$ related when $\alpha + \beta + \gamma = 0$?

(*b*) Prove that $(\alpha \times \beta) \times (\gamma \times \delta) = [\alpha\ \gamma\ \delta]\beta - [\beta\ \gamma\ \delta]\alpha$ where $[\alpha\ \beta\ \gamma]$ denotes the product $\alpha \,.\, (\beta \times \gamma)$.

Deduce that $[\alpha\ \beta\ \gamma]\delta = [\delta\ \beta\ \gamma]\alpha + [\delta\ \gamma\ \alpha]\beta + [\delta\ \alpha\ \beta]\gamma.$

1964 (Pass)

1. (*a*) Prove that $\alpha \times (\beta \times \gamma) = (\alpha \, . \, \gamma) \, \beta - (\alpha \, . \, \beta) \, \gamma$.

(*b*) If a straight line is equally inclined to three coplanar straight lines, show by vector methods, that it is perpendicular to their plane.

2. (*a*) Show by vector methods, that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

(*b*) Show by vector methods, that the line joining one vertex of a parallelogram to the middle point of an opposite side and the diagonal not passing through that vertex meet in a common point of trisection.

1964 (Honours)

1. (*a*) Prove by the method of vectors that the medians of a triangle meet in a point which trisects each of them.

(*b*) Establish the following formula for the derivative of the cross product of two vectors u and v which depend on a parameter t :

$$\frac{d}{dt}(u \times v) = u \times \frac{dv}{dt} + \frac{du}{dt} \times v.$$

If r be the position vector of a particle of mass m relative to a point O, F is the external force on the particle and M is the moment of F about O, show that $M = dH/dt$ where $H = r \times mv$, v is the velocity of the particle and t represents time.

1965 (Pass)

1. Show, by vector methods, that the line joining the mid-points of two sides of a triangle is parallel to and half the third side.

2. If α, β are two vectors, show that

$$(\alpha \times \beta)^2 + (\alpha \cdot \beta)^2 = \alpha^2 \beta^2.$$

3. If three distinct points A, B, C with position vectors α, β, γ respectively are collinear, show that there exist three non-zero numbers a, b, c such that

$$a + b + c = 0 \quad \text{and} \quad a\alpha + b\beta + c\gamma = \mathbf{0}.$$

1965 (Honours)

1. Define the vector and scalar products of two vectors with illustrative examples.

Given that each edge of a tetrahedron is equal to the edge opposite to it, prove that the lines which join the middle points of opposite edges are at right angles to those edges.

2. Define the terms divergence and curl of a vector.

Prove that (i) div curl $(X, Y, Z) = 0$,

(ii) curl curl $(X, Y, Z) =$ grad div (X, Y, Z)

$-$ div grad (X, Y, Z).

1966 (Pass)

1. (*a*) Show that $\alpha \times (\beta \times \gamma) = (\alpha \cdot \gamma)\, \beta - (\alpha \cdot \beta)\, \gamma$.

(*b*) If $\alpha = (-2, -2, 4)$, $\beta = (-2, 4, -2)$ and $\gamma = (4, -2, -2)$, calculate the value of $\alpha \cdot (\beta \times \gamma)$ in its simplest form and interpret your result geometrically.

2. (*a*) What do you mean by *vector product* of two vectors α and β? Find the unit vector perpendicular to each of the vectors $\alpha = 6i + 2j + 3k$ and $\beta = 3i - 6j - 2k$.

(*b*) Prove by vector method, the formula

$$a / \sin A = b / \sin B = c / \sin C$$

for any triangle ABC.

1966 (Honours)

1. (*a*) Prove by the method of vectors that the internal bisectors of the angles of a triangle are concurrent.

(*b*) If α be the position vector of a given point in space and ρ be the position vector of any point, find the locus of ρ if

(i) $(\rho-\alpha) \cdot \alpha=0$; (ii) $(\rho-\alpha) \cdot \rho=0$.

2. (*a*) Establish the following formula for the derivative of the dot product of two vectors α and β which depend on a parameter t :

$$\frac{d}{dt}(\alpha \cdot \beta)=\alpha \cdot \frac{d\beta}{dt}+\frac{d\alpha}{dt} \cdot \beta$$

If α has constant magnitude, show that α and $d\alpha/dt$ are perpendicular provided the magnitude of $d\alpha/dt$ is not zero.

(b) If two pairs of opposite edges of a tetrahedron are mutually perpendicular, prove that the third pair is also perpendicular to each other.